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Contents

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Abstract

Solonnikov[1] introduced a new system of the linear equations to treat the nonlinear problem obtained by the so called Hanzawa coordinate transformation to the free boundry problem for the Navier-Stokes equations in order to write it in a fixed domain. And, he proved the maximal regularity theorem in the L_2 Sobolev-Slobodetskii space in a bounded domain. In this paper, we prove the maximal $L_p - L_q$ regularity for the same linear problem as in Solonnikov[1] in uniformly C^3 domains under the assumption that the weak Dirichlet problem is uniquely solvable. Our approach is to construct R bounded solution operator for the generalized resolvent problem obtained by the Laplace transform with respet to time variable and to apply the Weis operator valued Fourier multiplier theorem.[2] The procedure in constructing the solution operator is similar to the theory of parameter elliptic problem.[3] There are two differences: one is to use the $\mathfrak{R}$ norm instead of the usual norm and another is to handle with the pressure term. Since the pressure term gives a non-local situation, in the localization the usual cut-off technique can not be used. To overcome this difficulty arising from the pressure term, we use the Grubb and Solonnikov technique[4,5] to eliminate the pressure term.

Keywords: maximal $L_p - L_q$ regularity, R bounded solution operator, uniform C^3 domain, finite covering space, the weak dirichlet problem

Introduction

Let Ω be a uniformly C^3 domain with boundary Γ in the N dimensional Euclidean space $\mathbb{R}^N$ $(N \ge 2)$, and let n be the unit outer normal to Γ. This paper deals with the linear problem:

$$\begin{cases} \partial_t v - \mathrm{D}iv(\mu D(v) - pI) = \mathbf{F}, \quad \mathrm{d}iv v = G = \mathrm{d}iv\mathbf{G} & \mathrm{in}\,\Omega \times (0,\mathrm{T}), \\ \partial_t \rho + A_\sigma \cdot \nabla_\Gamma \rho - v \cdot n + \mathcal{F}\, vP = D & \mathrm{on}\,\Gamma \times (0,\mathrm{T}), \\ (\mu D(v) - pI - ((\mathcal{B} + \delta\Delta_\Gamma)\rho)\mathbf{I})n = \mathbf{H} & \mathrm{on}\,\Gamma \times (0,\mathrm{T}), \\ (v,\rho)|_{t=0} = (v_0, \rho_0) & \mathrm{on}\,\Omega \times \Gamma. \end{cases} \tag{1.1}$$

The unknowns are the vector field $v = (v_1, \ldots, v_N)^{\mathrm{T}}$, where M^{T} denotes the transposed M, and the scalar functions $p(x,t)$ and $\rho(x,t)$, while $F, G, G, D \, and \, H$ are prescribed functions, and v_0 and ρ_0 are prescribed initial data. $D(v)$ is the doubled rate-of-strain tensor whose $(i,j)^{th}$ components are $D_{ij}(v) = \dfrac{\partial v_j}{\partial x_i} + \dfrac{\partial v_i}{\partial x_j}$; I the $N \times N$ identity matrix; $\mathcal{F}$ a bounded linear operator from $H_q^2(\Omega)$ into $H_q^1(\Omega)$; $\mathcal{B}$ a bounded linear operator from $W_q^{2-1/q}(\Gamma)$ into $W_q^{1-1/q}(\Gamma)$; Δ_Γ the Laplace-Beltrami operator on Γ; $\nabla_\Gamma = \nabla - n(n \cdot \nabla)$ the tangential gradient on Γ. Moreover, $\mu = \mu(x)$ and $\delta = \delta(x)$ are given C^1 functions defined on $\overline{\Omega}$, and $A_\sigma(x)$ is a vector field defined on Γ depending on a parameter $\sigma \in [0,1)$. Finally, for any matrix field K with $(i,j)^{th}$ components K_{ij}, the quantity $\mathrm{D}iv K$ denotes an N-vector with components $\sum_{j=1}^N \partial_j K_{ij}$, and also for any vector of functions $w = (w_1, \ldots, w_N)^{\mathrm{T}}$, we set $\mathrm{d}iv w = \sum_{j=1}^N \partial_j w_j$. Throughout the paper, we assume that

$$\begin{gathered} m_0 \le \mu(x), \delta(x) \le m_1, \quad |\nabla(\mu(x), \delta(x))| \le m_1 \quad \text{for all } x \in \overline{\Omega}, \\ \|\mathcal{F}\,(v)\|_{H_q^2(\Omega)} \le m_1 \|v\|_{H_q^1(\Omega)}, \quad \|\mathcal{B}_\rho\|_{W_q^{1-1/q}(\Gamma)} \le m_1 \|\rho\|_{W_q^{2-1/q}(\Gamma)} \end{gathered} \tag{1.2}$$

for some positive constants m_0 and m_1. Moreover, we assume that $A_0 = 0$ and that for any $\sigma \in (0,1)$, A_σ satisfies the assumptions:

$$|A_\sigma(x)| \le m_2, |A_\sigma(x) - A_\sigma(y)| \le m_2 |x-y|^a \quad \text{for any x,y} \in \Gamma, \quad \|A_\sigma\|_{W_r^{2-1/r}(\Gamma)} \le m_3 \sigma^{-b} \tag{1.3}$$

for some positive constants m_2, m_3, a and b that are independent of $\sigma \in (0,1)$, where r is an exponent with $N < r < \infty$. Notice that for $\sigma = 0$, the third equation in (1.1), the kinematic equation, reads

$$\partial_t \rho - v \cdot n + \mathcal{F}\, v = D \quad \text{on } \Gamma \times (0,\mathrm{T}).$$

Let $p = (p_1, \ldots, p_{N-1})$ be local coordinates on a surface $\Gamma' \subset \Gamma$ so that Γ' is represented by the equation $x = r(p)$. Let $g_{ij} = \dfrac{\partial r}{\partial p_i} \cdot \dfrac{\partial r}{\partial p_j}$, and let G be an $N \times N$ matrix with $(i,j)^{th}$ component g_{ij}, which is called the first fundamental form of Γ Let g^{ij} be the $(i,j)^{th}$ element of the inverse matrix G^{-1} of G and let $g = \sqrt{\det G}$. Then, the Laplace Beltrami operator Δ_Γ on Γ is defined by

$$\Delta_\Gamma f = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{N-1} \frac{\partial}{\partial p_i} (\sqrt{g} g^{ij} \frac{\partial}{\partial p_j} f(r(p))).$$

Moreover, $A_\sigma \cdot \nabla_\Gamma$ is represented on Γ' by

$$A_\sigma \cdot \nabla_\Gamma f = \sum_{i=1}^{N-1} A_{\sigma i} \frac{\partial}{\partial p_i} f(r(p))$$

with some $N-1$ vector $(A_{\sigma,1}, \ldots, A_{\sigma,N-1})$. We may assume that $A_\sigma = (A_{\sigma,1}, \ldots, A_{\sigma,N-1})$ is defined globally on Γ.

Problem (1.1) arises in the linearization of the time-dependent problem with a free boundary describing the evolution of viscous incompressible capillary fluid with a coefficient of surface tension δ. In fact, this problem is formulated as follows:

$$\begin{cases} v_t + (v \cdot \nabla)v - \mu\Delta v + \nabla p = 0, \quad \mathrm{div} v = 0 & x \in \Omega_t, \quad t \in (0,T), \\ (\mu D(v) - pI)n_t = \delta H n_t, \quad V_n = v \cdot n_t & x \in \Gamma_t, \quad t \in (0,T), \\ \Omega_t|_{t=0} = \Omega, \quad v(x,0) = v_0(x) & x \in \Omega. \end{cases} \tag{1.4}$$

Where, Ω_t is the evolution of the reference domain Ω at time $t > 0$, Γ_t the boundary of Ω_t, and n_t the unit outer normal to Γ_t. Moreover, V_n is the velocity of the evolution of the free surface Γ_t in the n_t direction and H is the $N-1$ times mean curvature of Γ_t. The equation $V_n = v \cdot n_t$ is the non-slip condition on the free surface. Since Ω_t is unknown, the so-called Hanzawa coordinate tranfromation is applied to write Eq. (1.4) in a fixed domain Ω. Namely, introducing the unknown function ρ, we represent Γ_t as

$$\Gamma_t = \{x = y + \rho(y,t)n \mid y \in \Gamma\} \quad (t > 0).$$

In this case, the kinetic condition $V_n = v \cdot n_t$ reads

$$\rho_t + <v \mid \nabla'_\Gamma \rho> - n \cdot v + O(|\nabla'_\Gamma \rho|^2 |v|) = 0 \quad \text{on } \Gamma \times (0,\mathrm{T})$$

Where, $<\cdot|\cdot>$ is the inner product on the tangent space of Γ and $O(|\nabla'_\Gamma \rho|^2|v|)$ denotes the nonlinear part of order 3. If we move the nonlinear term $<v \mid \nabla'_\Gamma \rho>$ to the right side and use the fixed point argument, then we can show the local well-posedness only in the small velocity case. To handle with the large velocity case, Solonnikov[1] introduced an approximation v_σ of initial data v_0, and then writing $<v \mid \nabla'_\Gamma \rho> = <v_\sigma \mid \nabla'_\Gamma \rho> + <v - v_\sigma \mid \nabla'_\Gamma \rho>$ we have the linearization principle for the kinetic equation as follows:

$$\rho_t + <v_\sigma \mid \nabla'_\gamma \rho> - n \cdot v = d(v,\rho). \quad \text{on } \Gamma \times (0,\mathrm{T})$$

with $d(v,\rho) = <v_\sigma - v \mid \nabla'_\Gamma \rho> + O(|\nabla'_\Gamma \rho|^2|v|)$. Thus, as a linearized problem of Eq. (1.4), we have (1.1), where $<v_\sigma \mid \nabla'_\Gamma \rho>$ is written as $A_\sigma \cdot \nabla_\Gamma \rho$ for $\sigma \in (0,1)$.

Problem (1.1) has been studied by Solonnikov[1] in anisotropic Sobolev-Slobodetskii spaces $W_2^{\ell,\ell/2}$ in a cylindrical domain $Q_T = \Omega \times (0,T)$ under the assumption that Ω is a bounded domain with a smooth boundary Γ. The purpose of this paper is to prove the maximal $L_p - L_q$ regularity of problem (1.1). Namely, the solutions v and ρ obtained in this paper belong to the following functional spaces:

$$\begin{aligned} v \quad &\in L_p((0,T), H_q^2(\Omega)^N) \cap H_p^1((0,T), L_q(\Omega)^N), \\ \rho \quad &\in L_p((0,T), H_q^3(\Omega)) \cap H_p^1((0,T), H_q^2(\Omega)). \end{aligned} \tag{1.5}$$

The maximal $L_p - L_q$ theory is a main tool to study the global well-posedness of free boundary problems for the Navier-Stokes equations in unbounded domains. In fact, in the unbounded domain case, only polynomial decay rates are obtained with suitable space norm pointwisely in time, which guarantees only global in time summability with rather large.[6,7]

As a related topics, the case that $A_\sigma = A_\sigma(x,t)$ has been treated by Prüss et al.,[8–10] in the maximal L_p regularity class and Shimizu et al.,[11] in the maximal $L_p - L_q$ regularity class. Their problem arises in the linearization of the time-dependent problem with a sharp interface describing the evolution of two different viscous incompressible capillary fluids. Of course, their approach is completely different from that in this paper, because they treated the time dependent coefficient case, while the coefficients in this paper is time independent.

We now state some references for free boundary problems for the Navier-Stokes equtions Eq. (1.4). The problem has been studied by many mathematicians in the following two cases: Ω is a bounded domain or a layer defined by $\{x = (x_1,\ldots,x_N) \in R^N \mid 0 < x_N < b\}$. The former is called a drop problem and the latter an ocean problem. When Ω is a layer, the local well-posedness was proved by Beal,[12] Allain[13] and Tani[14] in the L_2 Sobolev-Slobodetski space in the $\delta > 0$ case, and by Abels[15] in the L_p Sobolev-Slobodetski space in the $\delta = 0$ case. When Ω is a bounded domain, the local well-posendess is proved by Solonnikov[16,17] in the L_2 Sobolev-Slobodetski space, by Schweizer[18] in the semigroup setting, by Moglilevski[19] and Solonnikov[20] in the Hölder spaces in the $\delta > 0$ case, and by Solonnikov[21] and Mucha et al.,[22] in L_p Sobolev-Slobodetsk i space and by Shibata et al.,[23] in the maximal $L_p - L_q$ class in the $\delta = 0$ case. Recently, in the case where Ω is a uniformly C^3 domain and $\delta > 0$, Shibata[24] proved the local well-posedness in the maximal $L_p - L_q$ class under the assumption that the weak Dirichlet problem is uniquely solvable.

When Ω is a layer, the global well-posedness was proved by Beale[12] and Tani[25] in the $\delta > 0$ case, and by Sylvester et al.,[26] in the $\delta = 0$ case. The decay rate was studied by Beale et al.,[27] Sylvester[28], Hataya[29]. When Ω is a bounded domain, the global well-posedness was proved by Solonnikov[30] in the L_2 Sobolev-Slobodetski i space, by Padula et al.,[31] in the Hölder spaces, and by Shibata[32] in the L_p in time and L_q in space setting under the assumption that is Ω close to ball and initial data are small in the $\delta > 0$ case, and by Solonnikov[21] in the L_p Sobolev-Slobodetski i space and by Shibata[33] in the maxima $L_p - L_q$ class in the $\delta = 0$ case.

To prove the local well-posedness for large initial data, in the above references the Lagrange coordinate transformation was mostly used to transform the problem to the reference domain. But, if we apply the theory obtained in Shibata[34,35] to the problem obtained by the Hanzawa coordinate transformation, then we need the smallness assumption on initial velocity. To avoid this, it is necessary to use the linear problem introduced by Solonnikov.[1]

To prove the maximal $L_p - L_q$ regularity of Eq. (1.1), our main tool is to use R -bounded solution operators of the corresponding generalized resolvent problem (1.6) given below and the Weis operator valued Fourier multiplier theorem.[2] Thus, the main part of this paper is devoted to proving the existence of R bounded solution operators and the uniqueness of solutions of problem:

$$\begin{cases} \lambda u - \mathrm{Div}(\mu \mathbf{D}(\mathbf{u}) - q\mathbf{I}) = \mathbf{f}, \quad \mathrm{div}\,\mathbf{u} & = g = \mathrm{div}\, g \qquad \text{in}\,\Omega, \\ \lambda h + A_\sigma \cdot \nabla_\Gamma h - \mathbf{u} \cdot \mathbf{n} + F\, u & = d \qquad \text{on}\,\Gamma, \\ (\mu \mathbf{D}(u) - q\mathbf{I} - ((B + \delta \Delta_\Gamma) h)\mathbf{I})\mathbf{n} & = \mathbf{h} \qquad \text{on}\,\Gamma. \end{cases} \tag{1.6}$$

with complex parameter λ varying in $\Sigma_{\varepsilon,\lambda_0}$. Where, we have set

$$\Sigma_{\varepsilon,\lambda_0}=\{\lambda\in C\,\|\arg\lambda|\le\pi-\varepsilon,|\lambda|\ge\lambda_0\}.$$

To state the assumptions and main results of this paper, at this point we explain some further notation used throughout the paper. We denote the set of all complex numbers, real numbers and natural numbers by C, R, and N, respectively. We set $N_0=N\cup\{0\}$. For any multi-index $\alpha=(\alpha_1,\ldots,\alpha_N)\in N_0^N$, we set $|\alpha|=\sum_{j=1}^N\alpha_j$, and $\partial_x^\alpha=\partial_1^{\alpha_1}\cdots\partial_N^{\alpha_N}$ for $x=(x_1,\ldots,x_N)$ and $\partial_j=\partial/\partial x_j$. For scalar, θ, and N-vector, $u=(u_1,\ldots,u_N)^{\mathrm{T}}$, functions and $n\in N_0$, we set $\nabla^n\theta=(\partial_x^\alpha\theta\,\|\alpha|=n)$, $\nabla^n u=(\partial_x^\alpha u_j\,\|\alpha|=n,j=1,\ldots,N)$. In particular, $\nabla^0\theta=\theta$, $\nabla^1\theta=\nabla\theta$, $\nabla^0u=u$, and $\nabla^1u=\nabla u$. For any domain $G\subset R^N$, let $L_q(G)$, $H_q^m(G)$, and $B_{p,q}^s(G)$ be the standard Lebesgue, Sobolev, and Besov spaces on G, and let $\|\bullet\|_{L_q(G)}$, $\|\bullet\|_{H_q^m(G)}$, and $\|\bullet\|_{B_{p,q}^s(G)}$ denote their respective norms. In particular, we set $H_q^0(G)=L_q(G)$, $B_{pp}^s(G)=W_p^s(G)$, and $\|\bullet\|_{B_{pp}^s(G)}=\|\bullet\|_{W_p^s(G)}$. We use bold lowercase letters to denote N-vectors and bold capital letters to denote $N\times N$ matrices. For any N vector a, a_i denotes the i^{th} component of a and for any $N\times N$ matrix A, A_{ij} denotes the $(i,j)^{th}$ component of A, and moreover the $N\times N$ matrix whose $(i,j)^{th}$ component is K_{ij} is written as (K_{ij}). Let δ_{ij} be the Kronecker delta symbols, that is $\delta_{ii}=1$ and $\delta_{ij}=0$ for $i\ne j$. $I=(\delta_{ij})$ is the $N\times N$ identity matrix. For two $N\times N$ matrices $A=(a_{ij})$ and $B=(b_{ij})$, we write $A:B=\sum_{i,j=1}^N a_{ij}b_{ji}$. For any N-vectors a and b, let $a\cdot b=<a,b>=\sum_{i=1}^N a_ib_i$. For any N vector a, let $a_\tau=a-<a,n>n$. Given two Banach spaces X and Y, $X+Y=\{x+y\,|\,x\in X,y\in Y\}$, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from X into Y, and $\mathcal{L}(X,Y)$ is written simply by $\mathcal{L}(X)$. X^d denotes the d-product space of X, that is $X^d=\{x=(x_1,\ldots,x_d)\,|\,x_i\in X\}$, while the norm of X^d is simply written as $\|\bullet\|_X$, that is $\|f\|_X=\sum_{i=1}^d\|f_i\|_X$. For any domain $U\subset C$, $\mathrm{Hol}(U,\mathcal{L}(X,Y))$ denotes the set of all $\mathcal{L}(X,Y)$-valued holomorphic functions defined on U. Let $\mathcal{R}_{\mathcal{L}(X,Y)}(\{\mathcal{T}(\lambda)\,|\,\lambda\in U\})$ be the $\mathcal{R}$ bound of an operator family $\mathcal{T}(\lambda)\in\mathrm{Hol}(U,\mathcal{L}(X,Y))$. Let $\hat H_q^1(G)$ be a homogeneous space defined by $\hat H_q^1(G)=\{\theta\in L_{q,\mathrm{loc}}(G)\,|\,\nabla\theta\in L_q(G)^N\}$. Let $\hat H_{q,0}^1(G)$ and $H_{q,0}^1(G)$ be spaces defined by $X_{q,0}^1(G)=\{\theta\in X_q^1(G)\,|\,\theta|_{\partial G}=0\}$ for $X\in\{H,\hat H\}$, where ∂G denotes the boundary of G. Let $(u,v)_G=\int_G u\cdot\bar v\,dx$ and $(u,v)_{\partial G}=\int_{\partial G}u\cdot\bar v\,d\sigma$, where $\bar v$ denotes the complex conjugate of v and $d\sigma$ the surface element of ∂G. Let

$$J_q(G)=\{f\in L_q(G)^N\,|\,(f,\nabla\phi)_G=0\quad\text{for all}\,\phi\in\hat{\mathrm{H}}_{q',0}(\mathrm{G})\}.\qquad(1.7)$$

Let

$$C_{+,a}=\{\lambda\in C\,|\,\mathrm{Re}\lambda\ge a\},\quad\Sigma_\varepsilon=\{\lambda\in C\setminus\{0\}\,\|\arg\lambda|\le\pi-\varepsilon\},\quad\Sigma_{\varepsilon,a}=\{\lambda\in\Sigma_\varepsilon\,\|\lambda|\ge a\}.$$

For $1\le p\le\infty$, $L_p((a,b),X)$ and $H_p^m((a,b),X)$ denote the standard Lebesgue and Sobolev spaces of X-valued functions defined on an interval (a,b), and $\|\bullet\|_{L_p((a,b),X)}$ and $\|\bullet\|_{H_p^m((a,b),X)}$ denote their respective norms. For $\theta\in(0,1)$, Bessel potential spaces $H_p^\theta(\mathbb{R},X)$ are defined by

$$H_p^\theta(\mathbb{R},X)=\{f\in L_p(R,X)\,|\,\|f\|_{H_p^\theta(\mathbb{R},X)}=\left\|\mathcal{F}^{-1}[(1+\tau^2)^{\theta/2}\mathcal{F}[f](\tau)]\right\|_{L_p(\mathbb{R},X)}<\infty\}.$$

Here, $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the X valued Fourier transform and its inverse formula on $\mathbb{R}$. C denotes a generic constant and $C_{a,b,c,\ldots}$ denotes that the constant $C_{a,b,c,\ldots}$ depends on a,b,c,…The value of C and $C_{a,b,c,\ldots}$ may change from line to line.

We now introduce several definitions.

Definition 1.1: Let Ω be a domain in $\mathbb{R}^N$ with boundary $\partial\Omega$. We say that Ω is a uniformly C^3 domain, if there exist positive constants α,β and K such that for any $x_0=(x_{01},\ldots,x_{0N})\in\partial\Omega$ there exist a coordinate number j and a C^3 function $h(x')$ $(x'=(x_1,\ldots,\hat x_j,\ldots,x_N))$ defined on $B'_\alpha(x_{0'})$ with $x_{0'}=(x_{01},\ldots\hat x_{0j},\ldots,x_{0N})$ and $\|h\|_{H_\infty^3(B'_\alpha(x_{0'}))}\le K$ such that

$$\begin{aligned}\Omega\cap B_\beta(x_0)&=\{x\in\mathbb{R}^N\,|\,x_j>h(x')(x'\in B'_\alpha(x'_0))\}\cap B_\beta(x_0),\\ \partial\Omega\cap B_\beta(x_0)&=\{x\in\mathbb{R}^N\,|\,x_j=h(x')(x'\in B'_\alpha(x'_0))\}\cap B_\beta(x_0).\end{aligned}\qquad(1.8)$$

Here, $(x_1,\ldots,\hat x_j,\ldots,x_N)=(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_N)$, $B'_\alpha(x_{0'})=\{x'\in\mathbb{R}^{N-1}\,\|x'-x_{0'}|<\alpha\}$ and $B_\beta(x_0)=\{x\in\mathbb{R}^N\,\|x-x_0|<\beta\}$.

Definition 1.2: Let X and Y be two Banach spaces. A family of operators $\mathcal{T}\subset\mathcal{L}(X,Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X,Y)$, if there exist constants $C>0$ and $p\in[1,\infty)$ such that for each natural number n, $\{T_j\}_{j=1}^n\subset\mathcal{T}$, and $\{f_j\}_{j=1}^n\subset X$ there holds the inequality:

$$\left\|\sum_{j=1}^{n} r_j(u) T_j f_j\right\|_{L_p((0,1),Y)} \le C\left\|\sum_{j=1}^{n} r_j(u) f_j\right\|_{L_p((0,1),X)}. \quad (1.9)$$

The smallest such C is called R -bound of T on $L(X,Y)$, which is denoted by $R_{L(X,Y)}(T)$. Here the Rademacher functions r_k, $k\in N$, are given by $r_k:[0,1]\to\{-1,1\}, t\mapsto \mathrm{sign}(\sin(2^k\pi t))$.

Remark 1.3 The definition of R -boundedness is independent of $p\in[1,\infty)$ (cf. [36, p.26 3.2. Remarks (2)]).[36]

Definition 1.4: Let $1<q<\infty$. We say that the weak Dirichlet problem is uniquely solvable on $\hat{H}^1_{q,0}(\Omega)$, if the following assertion holds: For any $f\in L_q(\Omega)^N$ there exists a unique $\theta\in\hat{H}^1_{q,0}(\Omega)$ which satisfies the variational equation:

$$(\nabla\theta,\nabla\phi)_\Omega=(f,\nabla\phi)_\Omega \quad \text{for all } \phi\in \hat{\mathrm{H}}^1_{q',0}(\Omega), \qquad (1.10)$$

and the estimate: $\|\nabla\theta\|_{L_q(\Omega)}\le C_q\|f\|_{L_q(\Omega)}$ for some constant C_q independent of f,θ *and* ϕ, We define a bounded linear operator $K_0\in L(L_q(\Omega)^N,\hat{H}^1_{q,0}(\Omega))$ by letting $K_0(f)=\theta$.

Remark 1.5:

(1) Given $f\in L_q(\Omega)^N$ and $g\in W_q^{1-1/q}(\Gamma)$, there exists a unique $u\in H^1_q(\Omega)+\hat{H}^1_{q,0}(\Omega)$ that satisfies the variational equation:

$$(\nabla u,\nabla\phi)_\Omega=(f,\nabla\phi)_\Omega \quad \textit{for any } \phi\in\hat{\mathrm{H}}^1_{q',0}(\Omega) \qquad (1.11)$$

subject to $u=g$ on Γ. In fact, choosing $g\in W^1_q(\Omega)$ in such a way that $g|_\Gamma=g$, we see that $u=g+K_0(f-\nabla g)$ is a required function. Obviously, $\|\nabla u\|_{L_q(\Omega)}\le C_q(\|g\|_{W_q^{1-1/q}(\Gamma)}+\|f\|_{L_q(\Omega)})$. We define a linear operator K_1 by $K_1(f,g)=u$. In particular, $H^1_q(\Omega)+\hat{H}^1_{q,0}(\Omega)$ is the space for p in (1.1) and q in (1.6).

(2) In applications for our theorem stated below, it is important to prove the weak Dirichlet problem is uniquely solvable. For example, this holds for bounded domains, exterior domains, half-spaces, bent half-spaces, layer domains, tube domains, etc.

Since $H^1_q(\Omega)$ is usually not dense in $\hat{H}^1_q(\Omega)$, it does not hold that $\mathrm{div}\,\mathbf{u}=\mathrm{div}\,g$ implies $(\mathbf{u},\nabla\varphi)_\Omega=(g,\nabla\varphi)_\Omega$ for all $\varphi\in\hat{H}^1_q(\Omega)$. Of course, the opposite direction holds. Thus, finally we introduce the following definition.

Definition 1.6: For $\mathrm{u},\mathrm{g}\in L_q(\Omega)^N$, we say that $\mathrm{div}\mathbf{u}=\mathrm{div}g$ in Ω if there holds that $(u,\nabla\varphi)_\Omega=(g,\nabla\varphi)_\Omega$ for all $\varphi\in\hat{\mathrm{H}}^1_{q',0}$.

To solve the divergence equation $\mathrm{div}\,\mathbf{u}=g$ in Ω, it is necessary to assume that g is given by $g=\mathrm{div}\mathbf{g}$ for some $\mathbf{g}$, and so we define the space $DI_q(G)$ by

$$DI_q(G)=\{(g,\mathbf{g})\mid g\in H^1_q(G), g\in L_q(G), g=\mathrm{div}g\,\mathrm{in}\,\mathrm{G}\},$$

where G is any domain in $\mathbb{R}^N$.

We now state main results of this paper. We first state the existence theorems.

Theorem 1.7

Let $1<q<\infty$ and $0<\varepsilon<\pi/2$. Assume that the following conditions are satisfied:

Ω is a uniformly C^3 domain;

μ and δ are real valued functions satisfying the assumptions (1.2);

The weak Dirichlet problem is uniquely solvable on $\hat{H}^1_{q,0}(\Omega)$;

$A_0=0$ and A_σ is an $N-1$ vector of real valued functions with parameter $\sigma\in(0,1)$ satisfying (1.3).

Set

$$\begin{aligned} X_q(\Omega) &=\{(f,g,\mathbf{g},d,\mathbf{h})\mid \mathbf{f}\in L_q(\Omega)^N,(g,\mathbf{g})\in DI_q(\Omega), d\in W_q^{2-1/q}(\Gamma), h\in H^1_q(\Omega)^N\};\\ \mathcal{X}_q(\Omega) &=\{(F_1,\ldots,F_7)\mid F_1,F_3,F_7\in L_q(\Omega)^N, F_2\in W_q^{2-1/q}(\Gamma), F_4\in H^1_q(\Omega)^N,\\ &\quad F_5\in L_q(\Omega), F_6\in H^1_q(\Omega)\}; \end{aligned} \qquad (1.12)$$

$$\Lambda_{\sigma,\lambda_0}=\begin{cases}\Sigma_{\varepsilon,\lambda_0} & \mathrm{for}\,\sigma=0,\\ \mathbb{C}_{+,\lambda_0} & \mathrm{for}\,\sigma\in(0,1),\end{cases} \qquad \gamma_\sigma=\begin{cases}1 & \mathrm{for}\,\sigma=0,\\ \sigma^{-b} & \mathrm{for}\,\sigma\in(0,1).\end{cases}$$

Then, there exist a constant $\lambda_0 \ge 1$ and operator families:

$$\mathcal{A}(\lambda) \in \mathrm{Hol}(\Lambda_{\sigma,\lambda_0\gamma_\sigma}, \mathcal{L}(\mathcal{X}_q(\Omega), H_q^2(\Omega)^N)), \quad \mathcal{P}(\lambda) \in \mathrm{Hol}(\Lambda_{\sigma,\lambda_0\gamma_\sigma}, \mathcal{L}(\mathcal{X}_q(\Omega), H_q^1(\Omega) + \hat{H}_{q,0}^1(\Omega))),$$
$$\mathcal{H}(\lambda) \in \mathrm{Hol}(\Lambda_{\sigma,\lambda_0\gamma_\sigma}, \mathcal{L}(\mathcal{X}_q(\Omega), H_q^3(\Omega)))$$

such that for any $\lambda \in \Lambda_{\sigma,\lambda_0\gamma_\sigma}$ and $(f, g, g, d, h) \in X_q(\Omega)$,

$$u = \mathcal{A}(\lambda) F_\lambda(\mathbf{f}, g, \mathbf{g}, d, \mathbf{h}), q = \mathcal{P}(\lambda) F_\lambda(f, g, \mathbf{g}, d, h), h = \mathcal{H}(\lambda) F_\lambda(f, g, \mathbf{g}, d, h),$$

are solutions of (1.6), where

$$F_\lambda(f, g, g, d, h) = (f, d, \lambda^{1/2} h, h, \lambda^{1/2} g, g, \lambda g), \tag{1.13}$$

and

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), H_q^{2-j}(\Omega)^N)}(\{(\tau\partial_\tau)^\ell (\lambda^{j/2}\mathcal{A}(\lambda)) \mid \lambda \in \Lambda_{\sigma,\lambda_0\gamma_\sigma}\}) \le r_b,$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell \nabla\mathcal{P}(\lambda) \mid \lambda \in \Lambda_{\sigma,\lambda_0\gamma_\sigma}\}) \le r_b,$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), H_q^{3-k}(\Omega))}(\{(\tau\partial_\tau)^\ell (\lambda^k\mathcal{H}(\lambda)) \mid \lambda \in \Lambda_{\sigma,\lambda_0\gamma_\sigma}\}) \le r_b,$$

for $\ell = 0,1$, $j = 0,1,2$ and $k = 0,1$. Here, r_b is a constant depending on $m_0, m_1, m_2, m_3, K, \alpha, \beta, q$ *and* N but independent of $\sigma \in (0,1)$.

Remark 1.8: In this paper, $F_1, F_2, F_3, F_4, F_5, F_6$ *and* F_7 are corresponding variables to $f, d, \lambda^{1/2}h, h, \lambda^{1/2}g, g$ *and* λg respectively. The norm of space $X_q(\Omega)$ is given by

$$\|(F_1, \ldots, F_7)\|_{X_q(\Omega)} = \|(F_1, F_3, F_5, F_7)\|_{L_q(\Omega)} + \|F_2\|_{W_q^{2-1/q}(\Gamma)} + \|(F_4, F_6)\|_{W_q^1(\Omega)}.$$

Using Theorem 1.7 and the Weis operator valued Fourier multiplier theorem,[2] we have the following theorem.

Theorem 1.9: Let $1 < p, q < \infty$, and $T > 0$. Assume that $2/p + 1/q \ne 1$ and that the conditions i – iv stated in Theorem 1.7 are satisfied. Let $u_0 \in B_{q,p}^{2(1-1/p)}(\Omega)$ and $\rho_0 \in W_{q,p}^{3-1/p-1/q}(\Gamma)$ be initial data for problem (1.1) and let $\mathbf{F}$, G, $\mathbf{G}$,, D, and $\mathbf{H}$ be functions appearing in the right hand side of problem (1.1) with

$$F \in L_p((0,T), L_q(\Omega)^N)), \quad G \in L_p(\mathbb{R}, H_q^1(\Omega)) \cap H_p^{1/2}(R, L_q(\Omega)), \quad \mathbf{G} \in H_p^1(\mathbb{R}, L_q(\Omega)^N),$$
$$D \in L_p((0,T), W_q^{2-1/q}(\Gamma)), \quad \mathbf{H} \in L_p((\mathbb{R}, H_q^1(\Omega)^N) \cap H_p^{1/2}(\mathbb{R}, L_q(\Omega)^N).$$

We assume that u_0, $\mathbf{G}$, and $\mathbf{H}$ satisfy the following compatibility conditions:

$$div\ u_0 = div\ G|_{t=0} \ \ in\ \Omega.\ \ In\ addition,\ (\mu D(u_0)n)_\tau = (H|_{t=0})_\tau \text{on } \Gamma \ when\ \text{p}/2 + 1/\text{q} < 1. \tag{1.14}$$

Then, problem (1.1) admits solutions v, p *and* ρ with

$$v \in H_p^1((0,T), L_q(\Omega)^N) \cap L_p((0,T), H_q^2(\Omega)^N), \quad p \in L_p((0,T), H_q^1(\Omega) + \hat{H}_{q,0}^1(\Omega)),$$
$$\rho \in H_p^1((0,T), H_q^2(\Omega)) \cap L_p((0,T), H_q^3(\Omega)),$$

possessing the estimate:

$$\begin{aligned} &\|v\|_{L_p((0,T),H_q^2(\Omega))} + \|\partial_t v\|_{L_p((0,T),L_q(\Omega))} + \|\rho\|_{L_p((0,T),H_q^3(\Omega))} \\ &+ \|\partial_t\rho\|_{L_p((0,T),H_q^2(\Omega))} \le Ce^{c\gamma_\sigma T}(\|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \gamma_\sigma\|\rho_0\|_{B_{q,p}^{3-1/p-1/q}(\Gamma)} \\ &+ \|\mathbf{F}\|_{L_p((0,T),L_q(\Omega))} + \|D\|_{L_p((0,T),W_q^{2-1/q}(\Gamma))} + \|e^{-c\gamma_\sigma t}(G,\mathbf{H})\|_{L_p(R,H_q^1(\Omega))} \\ &+ \|e^{-c\gamma_\sigma t}(G,\mathbf{H})\|_{H_p^{1/2}(R,L_q(\Omega))} + \|e^{-c\gamma_\sigma t}\partial_t\mathbf{G}\|_{L_p(R,L_q(\Omega))}) \end{aligned} \tag{1.15}$$

for some positive constants C and c. Where, C and c in (1.15) are independent of $\sigma \in (0,1)$, and γ_σ is the number given in Theorem 1.7.

We next state the uniqueness theorems. In this paper, we say that the uniqueness holds for Eq. (1.1) if the following assertion is valid:

If v, p *and* ρ with

$$v \in L_p((0,T), H_q^2(\Omega)^N) \cap H_p^1((0,T), L_q(\Omega)^N), \quad p \in L_p((0,T), H_q^1(\Omega) + \hat{H}_{q,0}^1(\Omega)),$$
$$\rho \in L_p((0,T), W_q^{3-1/q}(\Gamma)) \cap H_p^1((0,T), W_q^{2-1/q}(\Gamma))$$

satisfy the homogeneous equations:

$$\begin{cases} \partial_t v - \mathrm{Div}(\mu\mathbf{D}(v) - pI) = 0, \quad \mathrm{div} v & = 0 & \text{in}\,\Omega\times(0,\mathrm{T}), \\ \partial_t\rho + A_\sigma\cdot\nabla_\Gamma\rho - v\cdot n + \mathcal{F}\,v & = 0 & \text{on}\,\Gamma\times(0,\mathrm{T}), \\ (\mu D(v) - pI - ((\mathcal{B}+\delta\Delta_\Gamma)\rho)\mathbf{I})\mathbf{n} & = 0 & \text{on}\,\Gamma\times(0,\mathrm{T}), \\ (v,\rho)|_{t=0} & = (0,0) & \text{on}\,\Omega\times\Gamma, \end{cases} \tag{1.16}$$

then, $v=0$ $p=0$, and $\rho=0$.

And also, we say that the uniqueness holds for Eq. (1.6) with U, where U is a subset of C, if the following assertion is valid:

Let $\lambda\in U$. If u, q and h with

$$\mathrm{u}\in H_q^2(\Omega), \quad q\in H_q^1(\Omega)+\hat{H}_{q,0}^1(\Omega), \quad h\in W_q^{3-1/q}(\Gamma)$$

satisfy the homogeneous equations:

$$\begin{cases} \lambda u - \mathrm{Div}(\mu\mathbf{D}(u) - q\mathbf{I}) = 0, \quad \mathrm{div} u & = 0 & \text{in }\Omega, \\ \lambda h + A_\sigma\cdot\nabla_\Gamma h - u\cdot n + \mathcal{F}\,u & = 0 & \text{on }\Gamma, \\ (\mu D(u) - qI - ((\mathcal{B}+\delta\Delta_\Gamma)h)I)n & = 0 & \text{on }\Gamma, \end{cases} \tag{1.17}$$

then $u=0$, $q=0$ and $h=0$.

In the case where $A_\sigma=0$, $\mathcal{F}\,u=0$, $\mathcal{B}\,h=0$, and δ is a positive number, the uniqueness for Eq. (1.6) follows from the existence theorem for the dual problem. Moreover, the uniqueness for Eq.(1.1) can be proved by applying the uniqueness theorem for Eq. (1.6) to the Laplace transform of solutions with respect to time variable. Thus, we have

Theorem 1.10: Assume that $A_\sigma=0$, $\mathcal{F}\,\mathrm{u}=0$, $\mathcal{B}\,h=0$, and δ is a positive constant, and that the conditions i and ii stated in Theorem 1.7 holds. In addition, we assume that the weak Dirichelet problem is uniquely solvable on $\hat{H}_{q',0}^1(\Omega)$ with $q'=q/(q-1)$. Then, the following assertions about uniqueness hold:

Let $1<q<\infty$ and $0<\varepsilon<\pi/2$. Then, there exists a $\lambda_0>0$ such that for any $\lambda\in\Sigma_{\varepsilon,\lambda_0}$, the uniqueness holds for Eq. (1.6).

Let $0<T<\infty$ and $1<p,q<\infty$. Then, the uniqueness holds for Eq. (1.1).

But, in the general case,, we do not have suitable dual problems, and so we prove the uniqueness by showing *a* priori estimates. In particular, we need a restriction on Ω.

Theorem 1.11: Assume that Ω is a uniformly C^3 domain whose inside is finitely covering, the definition of which will be given in Sect. 7 below. Moreover, we assume that the conditions ii–iv are satisfied. Then, the following assertions concerning the uniqueness hold.

Let $1<q<\infty$ and $0<\varepsilon<\pi/2$. Then, there exists a $\lambda_0>0$ such that for any $\lambda\in\Lambda_{\sigma,\lambda_0\gamma_\sigma}$, the uniqueness for Eq. (1.6) holds.

Let $0<T<\infty$ and $1<p,q<\infty$. Then, the uniqueness for Eq. (1.1) holds.

Remark 1.12: If Ω is a bounded domain, an exterior domain, a half space, a perturbed half space, a layer, a perturbed layer, and a tube, and if the boundary of Ω is a hypersurface of C^3 class, then Ω is a uniformly C^3 domain whose inside is finitely covering (cf. Example 7.2 in Sect. 7 below).

The paper is organized as follows: In Sect. 2, a reduced Stokes operator is introduced to eliminate the pressure term p from Eq. (1.1). And, we prove equivalence between Stokes operator and reduced Stokes operator. In Sect. 3, the existence of R bounded solution operators is proved for the model problem in $\mathbb{R}^N$. In Sect. 4, the existence of R bounded solution operators is proved for the model problem in $\mathbb{R}_+^N$. In Sect. 5, the existence of R bounded solution operator is proved in a bent half space. In Sect. 6, the existence of R bounded solution operator is proved in a uniform C^3 domain by constructing a parametrix. In Sect. 7, we prove a priori estimates of solutions to Eq. (1.6) and as a result, we obtain the uniqueness for Eq. (1.6). In Sect. 8, the maximal regularity theorem is proved by applying the Weis operator valued Fourier multiplier theorem to the representation formula of solutions to Eq. (1.1) obtained by using the R bounded solution operator. And also, the uniqueness for Eq. (1.1) is obtained by applying the uniqueness for Eq. (1.6) to the Laplace transform of Eq. (1.1) with respect to time variable. In Appendix A, a unique existence theorem for the weak Dirichlet problem is proved in $\mathbb{R}^N$ and $\mathbb{R}_+^N$. In Appendix B, the regularity theorem for the weak Dirichlet problem is proved. Notice that the uniqueness of strong solutions does not hold in general. In Appendix C, some Poincarés' type inequality is proved. Finally, in Appendix D, several properties of uniform C^3 domains are proved.

Reduced Stokes equations

Equivalence of stokes problem and reduced stokes problem

Since the pressure term p has no time evolution, we eliminate p and the divergence equation: $\mathrm{div}\mathbf{u} = g = \mathrm{div}\mathbf{g}$ following the idea due to Grubb et al.,4 Abels et al.[5] For this purpose, we introduce the reduced Stokes equations. Given $u \in H_q^2(\Omega)^N$ and $h \in W_q^{3-1/q}(\Gamma)$, let $K(u,h)$ be a unique solution of the weak Dirichlet problem:

$$(\nabla K(u,h),\nabla\varphi)_\Omega = (\mathrm{D}iv(\mu D(u)) - \nabla \mathrm{d}iv u,\nabla\varphi)_\Omega \quad \text{for any}\phi \in \hat{\mathrm{H}}^1_{\mathrm{q}',0}(\Omega) \tag{2.1}$$

subject to

$$K(u,h) =< \mu\mathrm{D}(u)n,n > -(B + \delta\Delta_\Gamma)h - \mathrm{div}\mathbf{u} \quad \text{on}\Gamma. \tag{2.2}$$

By Remark 1.5 1, we know the unique existence of $K(u,h) \in H_q^1(\Omega) + \hat{H}^1_{q,0}(\Omega)$ satisfying the estimate:

$$\|\nabla K(\mathrm{u},h)\|_{L_q(\Omega)} \le M_0(\|\nabla \mathrm{u}\|_{H_q^1(\Omega)} + \|h\|_{W_q^{3-1/q}(\Gamma)}) \tag{2.3}$$

for some constant $M_0 > 0$. We consider the reduced Stokes equations:

$$\begin{cases} \lambda u - \mathrm{D}iv(\mu D(u) - K(u,h)\mathbf{I}) = f & \text{in}\Omega, \\ \lambda h + A_\sigma \cdot \nabla_\Gamma h - u\cdot n + F\ u = d & \text{on}\Gamma, \\ (\mu D(u) - K(u,h)\mathbf{I} - ((B + \delta\Delta_\Gamma)h)\mathbf{I})n = h & \text{on}\Gamma. \end{cases} \tag{2.4}$$

Notice that the third condition in (2.4) is equivalent to

$$(\mu\mathrm{D}(\mathrm{u})n)_\tau = h_\tau \quad \text{and} \quad \mathrm{d}iv u = n\cdot h \quad \text{on } \Gamma. \tag{2.5}$$

In fact, by (2.2)

$$h\cdot n =< \mu D(\mathrm{u})n,n > -K(\mathrm{u},\rho) - (B + \delta\Delta_\Gamma)h = \mathrm{d}iv \mathrm{u} \quad \text{on } \Gamma.$$

We now discuss the equivalence between (1.6) and (2.4). We first assume that Eq. (1.6) is uniquely solvable. Let $f \in H_q^1(\Omega)^N$, $d \in W_q^{2-1/q}(\Gamma)$ and $h \in H_q^1(\Omega)^N$. Let $g \in H_q^1(\Omega)$ be a unique solution of the variational equation:

$$\lambda(g,\varphi)_\Omega + (\nabla g,\nabla\varphi)_\Omega = (-f,\nabla\varphi)_\Omega \quad \text{for any}\varphi \in \mathrm{H}^1_{\mathrm{q}',0}(\Omega) \tag{2.6}$$

subject to $g = \mathrm{n}\cdot\mathrm{h}$ on Γ. The unique existence of g is guaranteed for $\lambda \in \Sigma_{\varepsilon,\lambda_0}$ with large $\lambda_0 > 0$. From (2.6) it follows that

$$(g,\varphi)_\Omega = (-\lambda^{-1}(f + \nabla g),\nabla\varphi)_\Omega, \tag{2.7}$$

and so $\mathrm{d}iv\mathbf{g} = g$ with $\mathbf{g} = \lambda^{-1}(f + \nabla g)$. Let $u \in H_q^2(\Omega)^N$, $p \in H_q^1(\Omega) + \hat{H}^1_{q,0}(\Omega)$ and $h \in W_q^{3-1/q}(\Gamma)$ be unique solutions of Eq. (1.6). In view of Definition 1.6, we have

$$(u,\nabla\varphi)_\Omega = (g,\nabla\varphi)_\Omega \quad \text{for any}\varphi \in \hat{\mathrm{H}}^1_{\mathrm{q}',0}(\Omega). \tag{2.8}$$

Testing $\varphi \in \hat{H}^1_{q',0}(\Omega)$, from Eq. (1.6) we have

$$(f,\nabla\varphi)_\Omega = (\lambda u - \mathrm{Div}(\mu\mathrm{D}(\mathrm{u})) + \nabla q,\nabla\varphi)_\Omega = (\lambda\mathrm{u},\nabla\varphi)_\Omega - (\nabla\mathrm{d}iv\mathrm{u},\nabla\varphi)_\Omega + (\nabla(q - K(\mathrm{u},h)),\nabla\varphi)_\Omega.$$

Using $\mathrm{d}iv\,\mathrm{u} = g$ and (2.8), we have

$$(\lambda\mathrm{u},\nabla\varphi)_\Omega - (\nabla\mathrm{d}iv\mathrm{u},\nabla\varphi)_\Omega = \lambda(g,\nabla\varphi)_\Omega - (\nabla g,\nabla\varphi)_\Omega = (\mathrm{f},\nabla\varphi)_\Omega,$$

and so, we have

$$(\nabla(q - K(\mathrm{u},h)),\nabla\varphi)_\Omega = 0 \quad \text{for any}\varphi \in \hat{\mathrm{H}}^1_{\mathrm{q}',0}(\Omega).$$

Moreover, by (2.2), (2.6), and the boundary condition in Eq. (1.6)

$$q - K(u,h) =< \mu\mathrm{D}(u)n,n > -(B + \delta\Delta_\Gamma)\rho - \mathrm{n.h} - < \mu\mathrm{D}(\mathrm{u})\mathrm{n},\mathrm{n} > +(B + \delta\Delta_\gamma)h + \mathrm{d}iv\, u$$
$$= \mathrm{d}iv\, u - n\cdot h = g - g = 0$$

on Γ. Thus, the uniqueness implies that $q = K(u,h)$, which yields that u and h are solutions of Eq. (2.4).

Conversely, we assume that Eq. (2.4) is uniquely solvable. Let $\mathrm{f} \in L_q(\Omega)^N$, $(g,\mathbf{g}) \in DI_q(\Omega)$, $d \in W_q^{2-1/q}(\Gamma)$ and $h \in H_q^1(\Omega)^N$ in Eq. (1.6). Let $\theta \in H_q^1(\Omega) + \hat{H}^1_{q,0}(\Omega)$ be a unique solution of the weak Dirichlet problem:

$$(\nabla\theta,\nabla\varphi)_\Omega = (f,\nabla\varphi)_\Omega \quad \text{for any}\varphi \in \hat{\mathrm{H}}^1_{\mathrm{q}',0}(\Omega), \tag{2.9}$$

subject to $\theta = -\mathrm{n}\cdot\mathrm{h}$ on Γ, and then using θ we write Eq. (1.6) as

$$\begin{cases} \lambda u - \mathrm{Div}(\mu \mathrm{D}(u) - (q-\theta)\mathrm{I}) = f - \nabla\theta, \quad \mathrm{div} u & = g = \mathrm{div} g & \text{in } \Omega, \\ \lambda h + A_\sigma \cdot \nabla_\Gamma h - u \cdot n + F\ u & = d & \text{on } \Gamma, \\ (\mu \mathrm{D}(u) - (q-\theta)\mathrm{I} - ((B + \delta\Delta_\Gamma)h)\mathrm{I})\mathrm{n} & = h_\tau & \text{on } \Gamma. \end{cases} \tag{2.10}$$

Let $L \in H_q^1(\Omega) + \hat{H}_{q,0}^1(\Omega)$ be a unique solution of the weak Dirichlet problem:

$$(\nabla L, \nabla\varphi)_\Omega = (\lambda g - \nabla g, \nabla\varphi)_\Omega \quad \text{for any} \varphi \in \hat{\mathrm{H}}_{q',0}^1(\Omega) \tag{2.11}$$

subject to $L = -g$ on Γ. Let $u \in H_q^2(\Omega)$ and $h \in W_q^{3-1/q}(\Gamma)$ be unique solutions of the equations:

$$\begin{cases} \lambda u - \mathrm{Div}(\mu \mathrm{D}(u) - K(u,h)\mathrm{I}) = \mathrm{f} - \nabla\theta + \nabla L & \text{in } \Omega, \\ \lambda h + A_\sigma \cdot \nabla_\Gamma h - \mathrm{u} \cdot \mathrm{n} + F\ u = d & \text{on } \Gamma, \\ (\mu D(u) - K(u,h)I - ((B + \delta\Delta_\Gamma)h)I)n = \tilde{h} & \text{on } \Gamma, \end{cases} \tag{2.12}$$

where $\tilde{h} = h_\tau + gn$, that is $\tilde{h}_\tau = h_\tau$ and $\tilde{h} \cdot \mathrm{n} = g$. Testing $\varphi \in \hat{H}_{q',0}^1(\Omega)$ in Eq. (2.12) and using (2.9), we have

$$(\nabla L, \nabla\varphi)_\Omega = (\lambda u - \mathrm{Div}(\mu D(\mathrm{u}) - K(\mathrm{u},h)I), \nabla\varphi)_\Omega = \lambda(\mathrm{u}, \nabla\varphi)_\Omega - (\nabla \mathrm{div}\mathrm{u}, \nabla\varphi)_\Omega,$$

which, combined with (2.11), leads to

$$(\lambda g, \nabla\varphi)_\Omega - (\nabla g, \nabla\varphi)_\Omega = \lambda(u, \nabla\varphi) - (\nabla \mathrm{div} u, \nabla\varphi)_\Omega \quad \text{for any} \varphi \in \hat{\mathrm{H}}_{q',0}^1(\Omega). \tag{2.13}$$

Since $H_{q',0}^1(\Omega) \subset \hat{H}_{q',0}^1(\Omega)$, by (2.13) and $g = \mathrm{div}\ \mathbf{g}$, we have

$$\lambda(\mathrm{div}\mathrm{g} - \mathrm{div}\mathrm{u}, \varphi)_\Omega + (\nabla(\mathrm{div}\mathrm{g} - \mathrm{div}\varphi), \nabla\varphi)_\Omega = 0 \quad \text{for any} \varphi \in \mathrm{H}_{q',0}^1(\Omega).$$

Moreover, by (2.5) and $\tilde{h} \cdot \mathrm{n} = g$ on Γ, we have

$$\mathrm{div}\mathrm{u} - \mathrm{div}\mathrm{g} = \tilde{h} \cdot \mathrm{n} - g = 0 \quad \text{on } \Gamma.$$

Thus, the uniqueness of solutions for $\lambda \in \Sigma_{\varepsilon,\lambda_0}$ with large $\lambda_0 > 0$ yields that $g = \mathrm{div}\mathrm{g} = \mathrm{div}\mathrm{u}$. Thus, by (2.13) we have

$$(g, \nabla\varphi)_\Omega = (u, \nabla\varphi)_\Omega \quad \text{for any} \varphi \in \hat{\mathrm{H}}_{q',0}^1(\Omega)$$

whenever $\lambda \in \Sigma_{\varepsilon,\lambda_0}$. Thus, $\mathrm{div}\ \mathrm{u} = g = \mathrm{div}\mathrm{g}$ in Ω. By (2.12), (2.5) and (1.9)

$$(\nu \mathrm{D}(\mathrm{u})\mathrm{n})_\tau = \mathrm{h}_\tau,$$

$$\varepsilon, \lambda_0, m_0, m_1, q \ and\ N$$

Recalling that $g = -L$ (cf. (2.11)) and $\theta = -h \cdot n$ (cf. (2.9)), we have

$$n \cdot h = -\theta + L + \langle \mu \mathrm{D}(u)\mathrm{n}, n \rangle - K(\mathrm{u},h) - (B + \delta\Delta_\Gamma)h$$
$$= \langle \mu D(u)\mathrm{n}, n \rangle - (K(\mathrm{u},h) + \theta - L) - (B + \delta\Delta_\Gamma)h.$$

On the other hand, by (2.12)

$$\mathrm{f} = \lambda u - \mathrm{Div}(\mu \mathrm{D}(u) - (K(\mathrm{u},h) + \theta - L)\mathrm{I}) \quad \text{in } \Omega.$$

Thus, u, $p = K(\mathrm{u},h) + \theta - L$ and h are unique solutions of Eq. (1.1).

R-bounded solution operators for the reduced stokes equation

In the following, for the reduced Stokes equations (2.4) we prove the existence of R bounded solution operators as follows.

Theorem 2.1: Let $1 < q < \infty$ and $0 < \varepsilon < \pi/2$. Let $\Lambda_{\sigma,\lambda_0}$ be the set defined in Theorem 1.7. Assume that the conditions i–iv in Theorem 1.7 are satisfied. Set

$$\begin{aligned} Y_q(\Omega) &= \{(f,d,h) \quad | f \in L_q(\Omega)^N, \quad d \in W_q^{2-1/q}(\Gamma), h \in H_q^1(\Omega)^N\}, \\ \mathrm{Y}_q(\Omega) &= \{(F_1,\ldots,F_4) \quad | F_1, F_3 \in L_q(\Omega)^N, F_2 \in W_q^{2-1/q}(\Gamma), F_4 \in H_q^1(\Omega)^N\}. \end{aligned} \tag{2.14}$$

Then, there exist a constant $\lambda_* \ge 1$ and operator families:

$$A_r(\lambda) \in \mathrm{Hol}(\Lambda_{\sigma,\lambda_*\gamma_\sigma}, L(\mathrm{Y}_q(\Omega), H_q^2(\Omega)^N)), \quad H_r(\lambda) \in \mathrm{Hol}(\Lambda_{\sigma,\lambda_*\gamma_\sigma}, L(\mathrm{Y}_q(\Omega), H_q^3(\Omega)))$$

such that for any $\lambda \in \Lambda_{\sigma,\lambda_0\gamma_\sigma}$ and $(f,d,h) \in Y_q(\Omega)$,

$$u = A_r(\lambda)(f,d,\lambda^{1/2}h,h), \quad h = H_r(\lambda)(f,d,\lambda^{1/2},h),$$

are solutions of (2.4), and

$$R_{L(Y_q(\Omega),H_q^{2-j}(\Omega)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}A_r(\lambda))\mid\lambda\in\Lambda_{\sigma,\lambda*\gamma_\sigma}\})\le r_b,$$

$$R_{L(Y_q(\Omega),H_q^{3-k}(\Omega))}(\{(\tau\partial_\tau)^\ell(\lambda^k H_r(\lambda))\mid\lambda\in\Lambda_{\sigma,\lambda*\gamma_\sigma}\})\le r_b,$$

for $\ell=0,1$, $j=0,1,2$ and $k=0,1$. Here, r_b is a constant depending on $m_1, m_2, m_3, m_3, \lambda_0, p, q$ and N but independent of $\sigma\in(0,1)$, and γ_σ is the number defined in Theorem 1.7.

Remark 2.2 The norm of space $Y_q(\Omega)$ is defined by $\|(\mathrm{f},d,\mathrm{h})\|_{Y_q(\Omega)}=\|\mathrm{f}\|_{L_q(\Omega)}+\|d\|_{W_q^{2-1/q}(\Gamma)}+\|\mathrm{h}\|_{H_q^1(\Omega)}$; and the norm of space $Y_q(\Omega)$ is defined by

$\|(F_1,F_2,F_3,F_4)\|_{Y_q(\Omega)}=\|(F_1,F_3)\|_{L_q(\Omega)}+\|F_2\|_{W_q^{2-1/q}(\Gamma)}+\|F_4\|_{W_q^1(\Omega)}$.

Remark 2.3 As was pointed out in Subsec. 1.9, if u and h are solutions of Eq. (2.12), then u, h are also solutions of Eq. (??) with $p=K(u)+\theta-L$, and so Theorem 1.7 follows immediately from Theorem 2.1.

Model problem in $\mathbb{R}^N$

Constant μ case

In this subsection, we assume that μ is a constant satisfying the assumption (1.2), that is $m_0\le\mu\le m_1$. Given $u\in H_q^2(\mathbb{R}^N)^N$, let $u=K_0(u)$ be a unique solution of the weak Laplace problem:

$$(\nabla u,\nabla\varphi)_{\mathbb{R}^N}=(\mathrm{Div}(\mu(\mathrm{D(u)})-\nabla\mathrm{div}u,\nabla\varphi)_{\mathbb{R}^N}\quad\text{for any}\varphi\in\hat{\mathrm{H}}^1_{\mathrm{q}'}(\mathbb{R}^{\mathrm{N}}).\tag{3.1}$$

In this subsection, we consider the resolvent problem:

$$\lambda u-\mathrm{Div}(\mu\mathrm{D}(u)-K_0(\mathrm{u})\mathrm{I})=\mathrm{f}\quad\text{in }\mathbb{R}^{\mathrm{N}},\tag{3.2}$$

and prove the following theorem.

Theorem 3.1: Let $1<q<\infty$, $0<\varepsilon<\pi/2$, and $\lambda_0>0$. Then, there exists an operator family $A_0(\lambda)\in\mathrm{Hol}(\Sigma_{\varepsilon,\lambda_0},L(L_q(\mathbb{R}^N)^N,H_q^2(\mathbb{R}^N)^N))$ such that for any $\lambda=\gamma+i\tau\in\Sigma_{\varepsilon,\lambda_0}$ and $\mathrm{f}\in L_q(\mathbb{R}^N)^N$, $u=A_0(\lambda)f$ is a unique solution of Eq. (3.2) and

$$R_{L(L_q(\mathbb{R}^N)^N,H_q^{2-j}(\mathbb{R}^N)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}A_0(\lambda))\mid\lambda\in\Sigma_{\varepsilon,\lambda_0}\})\le r_b(\lambda_0)\tag{3.3}$$

for $\ell=0,1$ and $j=0,1,2$, where $r_b(\lambda_0)$ is a constant depending on $\varepsilon,\lambda_0,m_0,m_1,q$ and N, but independent of $\mu\in[m_0,m_1]$.

Proof: We first consider the Stokes equations:

$$\lambda\mathrm{u}-\mathrm{Div}(\mu(\mathrm{D(u)}-q\mathrm{I})=\mathrm{f},\quad \mathrm{div}\mathrm{u}=g=\mathrm{div}g\quad\text{in }\mathbb{R}^{\mathrm{N}}.\tag{3.4}$$

Since $\mathrm{Div}(\mu D(\mathrm{u})-q\mathrm{I})=\mu\Delta\mathrm{u}+\mu\nabla\mathrm{div}\mathrm{u}-\nabla q$, applying div to (3.4), we have

$$\lambda\,\mathrm{div}\mathrm{g}-2\mu\Delta g+\Delta q=\mathrm{div}\mathrm{f},$$

and so,

$$q=2\mu g+\Delta^{-1}(\mathrm{div}\mathrm{f}-\lambda\mathrm{div}g).$$

Combining this with (3.4) gives

$$\lambda\mathrm{u}-\mu\Delta\mathrm{u}=\mathrm{f}-\nabla\Delta^{-1}\mathrm{div}\mathrm{f}-\mu\nabla g+\lambda\nabla\Delta^{-1}\mathrm{div}\mathrm{g}.\tag{3.5}$$

We now look for a solution formula for Eq. (3.2). Let g be a solution of the variational problem:

$$(\lambda g,\varphi)_{\mathbb{R}^N}+(\nabla g,\nabla\varphi)_{\mathbb{R}^N}=(-\mathrm{f},\nabla\varphi)_{\mathbb{R}^N}\quad\text{for any}\varphi\in\hat{\mathrm{H}}^1_{\mathrm{q}'}(\mathbb{R}^{\mathrm{N}}),$$

and then this g is given by $g=(\lambda-\Delta)^{-1}\mathrm{div}f$. According to (2.7), we set $\mathrm{g}=\lambda^{-1}(f+\nabla\mathrm{g})$. Inserting these formulas into (3.5) gives

$$\lambda\mathrm{u}-\mu\Delta\mathrm{u}=f-(\mu-1)\nabla g=\mathrm{f}-(\mu-1)(\lambda-\Delta)^{-1}\nabla\mathrm{div}\mathrm{f}.$$

Thus, we have

$$u=F_\xi^{-1}\Big[\frac{F[\mathrm{f}](\xi)}{\lambda+\mu|\xi|^2}\Big]+(\mu-1)F_\xi^{-1}\Big[\frac{\xi\xi\cdot F[\mathrm{f}](\xi)}{(\lambda+\mu|\xi|^2)(\lambda+|\xi|^2)}\Big],$$

where F and F_ξ^{-1} denote the Fourier transform and its inversion formula defined by

$$\mathcal{F}[f](\xi)=\int_{R^N}e^{-ix\cdot\xi}f(x)dx,\quad \mathcal{F}_\xi^{-1}[g(\xi)](x)=\frac{1}{(2\pi)^N}\int_{R^N}e^{ix\cdot\xi}g(\xi)d\xi.$$

Thus, we define an operator family $A_0(\lambda)$ acting on $f\in L_q(\mathbb{R}^N)^N$ by

$$A_0(\lambda)f=\mathcal{F}_\xi^{-1}\Big[\frac{\mathcal{F}[f](\xi)}{\lambda+\mu|\xi|^2}\Big]+(\mu-1)\mathcal{F}_\xi^{-1}\Big[\frac{\xi\xi\cdot\mathcal{F}[f](\xi)}{(\lambda+\mu|\xi|^2)(\lambda+|\xi|^2)}\Big].$$

To prove the $\mathfrak{R}$ -boundedness of $A_0(\lambda)$, we use the following lemmas.

Lemma 3.2: Let $0<\varepsilon<\pi/2$. Then, for any $\lambda\in\Sigma_\varepsilon$ and $x\in[0,\infty)$, we have

$$|\lambda+x|\geq(\sin\frac{\varepsilon}{2})(|\lambda|+x). \tag{3.6}$$

Proof : Representing $\lambda=|\lambda|e^{i\theta}$ and using $\cos\theta\geq\cos(\pi-\varepsilon)=-\cos\varepsilon$ for $\lambda\in\Sigma_\varepsilon$, we have (3.6).

Lemma 3.3: Let $1<q<\infty$ and let U be a subset of C. Let $m=m(\lambda,\xi)$ be a function defined on $U\times(\mathbb{R}^N\setminus\{0\})$ which is infinitely differentiable with respect to $\xi\in\mathbb{R}^N\setminus\{0\}$ for each $\lambda\in U$. Assume that for any multi-index $\alpha\in\mathbb{N}_0^N$ there exists a constant C_α depending on α such that

$$|\partial_\xi^\alpha m(\lambda,\xi)|\leq C_\alpha|\xi|^{-|\alpha|} \tag{3.7}$$

for any $(\lambda,\xi)\in U\times(\mathbb{R}^N\setminus\{0\})$. Set

$$b(m)=\max_{|\alpha|\leq N+1}C_\alpha.$$

Let K_λ be an operator defined by

$$K_\lambda f=\mathcal{F}_\xi^{-1}[m(\lambda,\xi)\mathcal{F}[f](\xi)].$$

Then, the operator family $\{K_\lambda\mid\lambda\in U\}$ is $\mathcal{R}$ -bounded on $\mathcal{L}(L_q(\mathbb{R}^N))$ and

$$\mathcal{R}_{\mathcal{L}(L_q(R^N))}(\{K_\lambda\mid\lambda\in U\})\leq C_{N,q}b(m)$$

for some constant $C_{q,N}$ depending solely on q and N.

Proof: Lemma 3.3 was proved by Enomoto et al.[37] and Denk et al.[38] By Lemma 3.2, we have

$$|\partial_\xi^\alpha\frac{\lambda^{j/2}\xi^\beta}{\lambda+\mu|\xi|^2}|\leq C_\alpha|\xi|^{-|\alpha|}\lambda_0^{k/2},\quad |\partial_\xi^\alpha\frac{\xi_\ell\xi_m\lambda^{j/2}\xi^\beta}{(\lambda+\mu|\xi|^2)(\lambda+|\xi|^2)}|\leq C_\alpha|\xi|^{-|\alpha|}\lambda_0^{-k/2}\quad(\ell,m=1,\ldots,N)$$

for any $j\in\mathbb{N}_0$, $k\in\mathbb{N}_0$ and $\beta\in\mathbb{N}_0^N$ such that $j+k+|\beta|=2$ and for any $\alpha\in\mathbb{N}_0^N$ and $(\lambda,\xi)\in\Sigma_{\varepsilon,\lambda_0}\times(\mathbb{R}^N\setminus\{0\})$. Thus, by Lemma 3.3, we have (3.3), which completes the proof of Theorem 3.1. We conclude this section by introducing some fundamental properties of $\mathcal{R}$ -bounded operators and Bourgain's results concerning Fourier multiplier theorems with scalar multiplieres.

Proposition 3.4

a) Let X and Y be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$ -bounded families in $\mathcal{L}(X,Y)$. Then, $\mathcal{T}+\mathcal{S}=\{T+S\mid T\in\mathcal{T},S\in\mathcal{S}\}$ is also an $\mathcal{R}$ -bounded family in $\mathcal{L}(X,Y)$ and $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}+\mathcal{S})\leq\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})+\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S})$.

b) Let X, Y and Z be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$ -bounded families in $\mathcal{L}(X,Y)$ and $\mathcal{L}(Y,Z)$, respectively. Then, $\mathcal{ST}=\{ST\mid T\in\mathcal{T},S\in\mathcal{S}\}$ also an $\mathcal{R}$ -bounded family in $\mathcal{L}(X,Z)$ and $\mathcal{R}_{\mathcal{L}(X,Z)}(\mathcal{ST})\leq\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})\mathcal{R}_{\mathcal{L}(Y,Z)}(\mathcal{S})$.

c) Let $1<p,q<\infty$ and let D be a domain in $\mathbb{R}^N$. Let $m=m(\lambda)$ be a bounded function defined on a subset U of $\mathbb{C}$ and let $M_m(\lambda)$ be a map defined by $M_m(\lambda)f=m(\lambda)f$ for any $f\in L_q(D)$. Then, $\mathcal{R}_{\mathcal{L}(L_q(D))}(\{M_m(\lambda)\mid\lambda\in U\})\leq C_{N,q,D}\|m\|_{L_\infty(U)}$.

d) Let $n=n(\tau)$ be a C^1-function defined on $R\setminus\{0\}$ that satisfies the conditions $|n(\tau)|\leq\gamma$ and $|\tau n'(\tau)|\leq\gamma$ with some constant $c>0$ for any $\tau\in R\setminus\{0\}$. Let T_n be an operator-valued Fourier multiplier defined by $T_nf=\mathcal{F}^{-1}(n\mathcal{F}[f])$ for any f with $\mathcal{F}[f]\in\mathcal{D}(\mathbb{R},L_q(D))$. Then, T_n is extended to a bounded linear operator from $L_p(\mathbb{R},L_q(D))$ into itself. Moreover, denoting this extension also by T_n, we have $\|T_n\|_{\mathcal{L}(L_p(R,L_q(D)))}\leq C_{p,q,D}\gamma$.

Proof: The assertions a) and b) follow from [36, p.28, Proposition 3.4], and the assertions c) and d) follow from [36, p.27, Remarks 3.2].[36,39]

Perturbed problem in $\mathbb{R}^N$

In this subsection, we consider the case where $\mu(x)$ is a real valued funtion satisfying (1.2). Let x_0 be any point in Ω and let d_0 be a positive number such that $B_{d_0}(x_0)\subset\Omega$. In view of (1.2), we assume that

$$|\mu(x)-\mu(x_0)|\leq m_1M_1\quad\text{for } x\in B_{d_0}(x_0), \tag{3.8}$$

where we have set $M_1 = d_0$. We assume that $M_1 \in (0,1)$ below. Let φ be a function in $C_0^\infty(\mathbb{R}^N)$ which equals 1 for $x \in B_{d_0/2}(x_0)$ and 0 outside of $B_{d_0}(x_0)$. Let

$$\tilde{\mu}(x) = \phi(x)\mu(x) + (1-\phi(x))\mu(x_0). \tag{3.9}$$

Let $\tilde{K}_0(u) \in \hat{H}_q^1(\mathbb{R}^N)$ be a unique solution of the weak Laplace problem:

$$(\nabla u, \nabla\varphi)_{\mathbb{R}^N} = (\mathrm{D}iv(\tilde{\mu}D(\mathrm{u})) - \nabla \mathrm{d}iv\mathrm{u}, \nabla\varphi)_{\mathbb{R}^N} \quad \text{for any} \varphi \in \hat{\mathrm{H}}^1_{\mathrm{q}'}(\mathbb{R}^{\mathrm{N}}). \tag{3.10}$$

We consider the resolvent problem:

$$\lambda u - \mathrm{D}iv(\tilde{\mu}\mathrm{D}(\mathrm{u}) - \tilde{K}_0(\mathrm{u})\mathrm{I}) = f \quad \text{in } \mathbb{R}^{\mathrm{N}}. \tag{3.11}$$

We shall prove the following theorem.

Theorem 3.5: Let $1 < q < \infty$ and $0 < \varepsilon < \pi/2$. Then, there exist $M_1 \in (0,1)$, $\lambda_0 \ge 1$ and an operator family $\tilde{A_0}(\lambda)$ with

$$\tilde{A_0}(\lambda) \in \mathrm{H}ol(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(L_q(\mathbb{R}^N)^N, H_q^2(\mathbb{R}^N)^N))$$

such that for any $\lambda \in \Sigma_{\varepsilon,\lambda_0}$ and $\mathrm{f} \in L_q(\mathbb{R}^N)^N$, $u = \tilde{A}(\lambda) f$ is a unique solution of Eq. (3.11), and

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^N, H_q^{2-j}(\mathbb{R}^N)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{A}_0(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \le \tilde{r}_b$$

for $\ell = 0,1$ and $j = 0,1,2$. Where, $\tilde{r}_b$ is a constant independent of M_1 and λ_0.

Proof : Let $u = K_{x_0}(u) \in \hat{H}_q^1(\mathbb{R}^N)$ be a unique solution of the weak Laplace equation:

$$(\nabla \mathrm{u}, \nabla\varphi)_{\mathbb{R}^N} = (\mathrm{D}iv(\mu(x_0)\mathrm{D}(u) - \nabla \mathrm{d}iv u, \nabla\varphi)_{\mathbb{R}^N} \quad \text{for any} \varphi \in \hat{\mathrm{H}}^1_{\mathrm{q}'}(\mathbb{R}^{\mathrm{N}}). \tag{3.12}$$

We consider the resolvent problem:

$$\lambda u - \mathrm{D}iv(\mu(x_0)\mathrm{D}(\mathrm{u}) - K_{x_0}(u)\mathrm{I}) = \mathrm{f} \quad \text{in } \mathbb{R}^{\mathrm{N}}. \tag{3.13}$$

Let $\mathcal{B}_{x_0}(\lambda) \in \mathrm{H}ol(\Sigma_{\varepsilon,1}, \mathcal{L}(L_p(\mathbb{R}^N)^N, H_q^2(\mathbb{R}^N)^N))$ be a solution operator of Eq. (3.13) such that for any $\lambda \in \Sigma_{\varepsilon,1}$ and $\mathrm{f} \in L_q(\mathbb{R}^N)^N$, $u = \mathcal{B}_{x_0}(\lambda) f$ is a unique solution of Eq.(3.13) and

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^N, H_q^{2-j}(\mathbb{R}^N)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{B}_{x_0}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,1}\}) \le \gamma_0 \tag{3.14}$$

for $\ell = 0,1$ and $j = 0,1,2$, where γ_0 is a constant independent of M_1 and $\nabla\phi$. Such an operator is given in Theorem 3.1 with $\mu = \mu(x_0)$ and $\lambda_0 = 1$. Inserting the formula: $u = \mathcal{B}_{x_0}(\lambda) f$ into (3.11) gives

$$\lambda u - \mathrm{D}iv(\tilde{\mu}(x)\mathrm{D}(\mathrm{u}) - \tilde{K}_0(\mathrm{u})\mathrm{I}) = \mathrm{f} - \mathcal{R}(\lambda) f \quad \text{in } \mathbb{R}^{\mathrm{N}}, \tag{3.15}$$

where we have set

$$\begin{aligned} \mathcal{R}(\lambda)\mathrm{f} &= \mathrm{D}iv(\tilde{\mu}(x)\mathrm{D}(\mathcal{B}_{x_0}(\lambda)\mathrm{f}) - \mu(x_0)\mathrm{D}(\mathcal{B}_{x_0}(\lambda)\mathrm{f})) \\ &\quad - \nabla(\tilde{K}_0(\mathcal{B}_{x_0}(\lambda) f) - K_{x_0}(\mathcal{B}_{x_0}(\lambda) f)). \end{aligned} \tag{3.16}$$

We shall estimate $\mathcal{R}(\lambda) f$. For any $\varphi \in \hat{H}_{q'}^1(\mathbb{R}^N)$, by (3.10) and (3.12), we have

$$(\nabla(\tilde{K}_0(\mathcal{B}_{x_0}(\lambda) f) - K_{x_0}(\mathcal{B}_{x_0}(\lambda) f)), \nabla\varphi)_{R^N} = ((\mathrm{D}iv((\tilde{\mu}(x) - \mu(x_0))\mathrm{D}(\mathcal{B}_{x_0}(\lambda)\mathrm{f})), \nabla\varphi)\mathbb{R}_{R^N}.$$

Since $\tilde{\mu}(x) - \mu(x_0) = \varphi(x)(\mu(x) - \mu(x_0))$, by (3.8) and (1.2), we have

$$\|\mathrm{D}iv((\tilde{\mu}(x) - \mu(x_0))D(\mathcal{B}_{x_0}(\lambda)\mathrm{f})\|_{L_q(\mathbb{R}^N)} \le M_1 \|\nabla^2 \mathcal{B}_{x_0}(\lambda) f\|_{L_q(\mathbb{R}^N)} + C_{m_1,\nabla\phi} \|\nabla \mathcal{B}_{x_0}(\lambda)\mathrm{f}\|_{L_q(\mathbb{R}^N)}.$$

Here and in the following, $C_{m_1,\nabla\varphi}$ denotes a generic constant depending on m_1 and $\|\nabla\phi\|_{L_\infty(R^N)}$. Thus, we have

$$\|\mathcal{R}(\lambda) f\|_{L_q(\mathbb{R}^N)} \le CM_1 \|\nabla^2 \mathcal{B}_{x_0}(\lambda) f\|_{L_q(\mathbb{R}^N)} + C_{m_1,\nabla\varphi} \|\nabla \mathcal{B}_{x_0}(\lambda)\mathrm{f}\|_{L_q(\mathbb{R}^N)}. \tag{3.17}$$

Here and in the following, C denotes a generic constants independent of M_1, m_1, and $\|\nabla\varphi\|_{L_\infty(\mathbb{R}^N)}$. Let λ_0 be any number ≥ 1 and let $n \in \mathbb{N}$, $\{\lambda_k\}_{k=1}^n \subset (\Sigma_{\varepsilon,\lambda_0})^n$, and $\{F_k\}_{k=1}^n \subset (L_q(\mathbb{R}^N)^N)^n$. By (3.17), (3.14) and Proposition 3.4, we have

$$\begin{aligned} &\int_0^1 \left\| \sum_{k=1}^n r_k(u) \mathcal{R}(\lambda_k) f_k \right\|_{L_q(\mathbb{R}^N)}^q du \\ &\le 2^{q-1} M_1^q \int_0^1 \left\| \sum_{k=1}^n r_k(u) \nabla^2 \mathcal{B}_{x_0}(\lambda_k) f_k \right\|_{L_q(\mathbb{R}^N)}^q du \\ &+ 2^{q-1} C_{m_1,\nabla\varphi}^q \int_0^1 \left\| \sum_{k=1}^n r_k(u) \nabla \mathcal{B}_{x_0}(\lambda_k) f_k \right\|_{L_q(\mathbb{R}^N)}^q du \end{aligned}$$

$$\leq 2^{q-1} M_1^q \int_0^1 \left\| \sum_{k=1}^n r_k(u) \nabla^2 B_{x0}(\lambda_k) \mathrm{f}_k \right\|_{L_q(\mathbb{R}^N)}^q du$$

$$+2^{q-1} C_{m_1,\nabla\varphi}^q \lambda_0^{-q/2} \int_0^1 \left\| \sum_{k=1}^n r_k(u) \lambda_k^{1/2} \nabla B_{x0}(\lambda_k) f_k \right\|_{L_q(\mathbb{R}^N)}^q du$$

$$\leq 2^{q-1}(M_1^q + C_{m_1,\nabla\varphi}^q \lambda_0^{-q/2}) \gamma_0^q \int_0^1 \left\| \sum_{k=1}^n r_k(u) \mathrm{f}_k \right\|_{L_q(\mathbb{R}^N)}^q du.$$

Choosing M_1 so small that $2^{q-1} M_1^q \gamma_0^q \leq (1/2)(1/q)^q$ and $\lambda_0 \geq 1$ so large that $2^{q-1} C_{m_1,\nabla\varphi}^q \gamma_0^q \lambda_0^{-q/2} \leq (1/2)(1/2)^q$, we have

$$R_{L(L_q(R^N))}(\{R(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq 1/2.$$

Analogously, we have

$$R_{L(L_q(R^N))}(\{\tau\partial_\tau R(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq 1/2.$$

Thus, $(I - R(\lambda))^{-1} = I + \sum_{j=1}^{\infty} R(\lambda)^j$ exists and

$$R_{L(L_q(R^N))}(\{(\tau\partial_\tau)^\ell (I - R(\lambda))^{-1} \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq 4 \quad \text{for } \ell=0,1. \tag{3.18}$$

Setting $\tilde{A_0}(\lambda) = B_{x_0}(\lambda)(I - R(\lambda))^{-1}$, by (3.14), (3.18) and Propsoition 3.4, we see that $\tilde{R_0}(\lambda)$ is a solution operator satisfying the required properties with $\tilde{r}_b = 4\gamma_0$.

To prove the uniqueness of solutions of Eq. (3.11), let $u \in H_q^2(\mathbb{R}^N)^N$ be a solution of the homogeneous equatuion:

$$\lambda u - \mathrm{Div}(\tilde{\mu}\mathrm{D}(u) - \tilde{K}_0(u)\mathrm{I}) = 0 \quad \text{in } \mathbb{R}^{\mathrm{N}}.$$

And then, u satisfies the non-homogeneous equation:

$$\lambda u - \mathrm{Div}(\mu(x_0)\mathrm{D}(u) - K_{x0}(u)\mathrm{I}) = Ru \quad \text{in } \mathbb{R}^{\mathrm{N}}, \tag{3.19}$$

where we have set

$$Ru = -\mathrm{Div}((\tilde{\mu}(\mathrm{x}) - \mu(x_0))\mathrm{D}(\mathrm{u})) + \nabla(\tilde{K}_0(\mathrm{u}) - K_{x0}(\mathrm{u})).$$

Analogously to the proof of (3.8), we have

$$\|Ru\|_{L_q(\mathbb{R}^N)} \leq CM_1 \|\nabla^2 u\|_{L_q(\mathbb{R}^N)} + C_{m_1,\nabla\phi} \|\nabla u\|_{L_q(\mathbb{R}^N)}. \tag{3.20}$$

On the other hand, applying Theorem 3.1 to (3.19) for $\lambda \in \Sigma_{\varepsilon,1}$, we have

$$|\lambda| \|u\|_{L_q(\mathbb{R}^N)} + |\lambda|^{1/2} \|u\|_{H_q^1(\mathbb{R}^N)} + \|u\|_{H_q^2(\mathbb{R}^N)} \leq C \|Ru\|_{L_q(\mathbb{R}^N)}. \tag{3.21}$$

Combining (3.20) and (3.21) gives

$$(\lambda_0^{1/2} - CC_{m_1,\nabla\varphi}) \|u\|_{H_q^1(\mathbb{R}^N)} + (1 - CM_1) \|\mathrm{u}\|_{H_q^2(\mathbb{R}^N)} \leq 0.$$

Choosing $M_1 \in (0,1)$ so small that $1 - CM_1 > 0$ and $\lambda_0 \geq 1$ so large that $\lambda_0^{1/2} - CC_{m_1,\nabla\varphi} > 0$, we have $u = 0$. This proves the uniqueness, and therefore we have proved Theorem 3.5

Model problem in $\mathbb{R}_+^N$

In this section, we assume that μ , δ , and A_σ ($\sigma \in [0,1)$) are constants and an $N-1$ constant vector satisfying the conditions:

$$m_0 \leq \mu, \delta \leq m_1, \quad A_0 = 0, \quad |A_\sigma| \leq m_2(\sigma \in (0,1)). \tag{4.1}$$

Let

$$\mathbb{R}_+^N = \{(x_1,\ldots,x_N) \in \mathbb{R}^N \mid x_N > 0\}, \quad \mathbb{R}_0^N = \{(x_1,\ldots,x_N) \in \mathbb{R}^N \mid x_N = 0\}, \quad n_0 = (0,\ldots,0,-1).$$

Given $u \in H_q^2(\mathbb{R}_+^N)^N$ and $h \in W_q^{3-1/q}(\mathbb{R}_0^N)$, let $K_0(u,h) \in H_q^1(\mathbb{R}_+^N) + \hat{H}_{q,0}^1(\mathbb{R}_+^N)$ be a unique solution of the weak Dirichlet problem:

$$(\nabla K_0(u,h), \nabla\varphi)_{\mathbb{R}_+^N} = (\mathrm{Div}(\mu D(u)) - \nabla \mathrm{div} u, \nabla\varphi)_{\mathbb{R}_+^N} \quad \text{for any} \varphi \in \hat{\mathrm{H}}_{q',0}^1(\mathbb{R}^{\mathrm{N}}), \tag{4.2}$$

subject to $K_0(u,h) =< \mu\mathrm{D}(\mathrm{u})n_0, n_0 > -\delta\Delta' h - \mathrm{div} u$ on $\mathbb{R}_0^N$, where $\Delta' h = \sum_{j=1}^{N-1} \partial^2 h / \partial x_j^2$. In this section, we consider the half space problem:

$$\begin{cases} \lambda u - Div(\mu D(u) - K_0(u,h)I) & = f & \text{in } \mathbb{R}^N_+, \\ \lambda h + A_\sigma \cdot \nabla' h - u \cdot n_0 & = d & \text{on } \mathbb{R}^N_0, \\ (\mu D(u) - K_0(u,h)I)n_0 - \delta(\Delta' h)n_0 & = h & \text{on } \mathbb{R}^N_0, \end{cases} \tag{4.3}$$

where $\nabla' = (\partial_1, \ldots, \partial_{N-1})$. The last equations in (4.3) are equivalent to

$$(\mu D(u)n_0)_\tau = h_\tau \quad \text{and} \quad div u = h \cdot n_0 \quad \text{on } \mathbb{R}^N_0.$$

Where, we have set $h_\tau = h - < h, n_0 > n_0$. . We shall show the following theorem

Theorem 4.1: Let $1 < q < \infty$, let μ, δ, and A_σ are constants and an $N-1$ constant vector satisfying the conditions in (4.1). Let $\Lambda_{\sigma,\lambda_0}$ the set defined in Theorem 1.7. Assume that the conditions in (4.1) hold. Let $Y_q(\mathbb{R}^N_+)$ and $Y_q(\mathbb{R}^N_+)$ be spaces defined by replacing Ω and Γ by $\mathbb{R}^N_+$ and $\mathbb{R}^N_0$ in (2.14). Then, there exist a constant $\lambda_0 \geq 1$ and operator families:

$$A_0(\lambda) \in Hol(\Lambda_{\sigma,\lambda_0}, L(Y_q(R^N_+), H^2_q(R^N_+)^N)), \quad H_0(\lambda) \in Hol(\Lambda_{\sigma,\lambda_0}, L(Y_q(\mathbb{R}^N_+), H^3_q(\mathbb{R}^N_+))) \tag{4.4}$$

such that for any $\lambda = \gamma + i\tau \in \Lambda_{\sigma,\lambda_0}$ and $(f,d,h) \in Y_q(\mathbb{R}^N_+)$,

$$u = A_0(\lambda)(f,d,\lambda^{1/2}h,h), \quad h = H_0(\lambda)(f,d,\lambda^{1/2},h),$$

are unique solutions of (4.3), and

$$\begin{aligned} &R_{L(Y_q(\mathbb{R}^N_+), H^{2-j}_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}A_0(\lambda)) \mid \lambda \in \Lambda_{\sigma,\lambda_0}\}) \leq r_b, \\ &R_{L(Y_q(\mathbb{R}^N_+), H^{3-k}_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^\ell(\lambda^k H_0(\lambda)) \mid \lambda \in \Lambda_{\sigma,\lambda_0}\}) \leq r_b, \end{aligned} \tag{4.5}$$

for $\ell = 0,1$, $j = 0,1,2$ and $k = 0,1$. Here, r_b is a constant depending on m_0, m_1, m_2, λ_0, q, and N.

Remark 4.2: In this section, what the constant depends on m_0, m_1, m_2 means that the constant c depends on m_0, m_1, m_2 but is independent of μ, δ and A_σ whenever $\mu \in [m_0, m_1]$, $\delta \in [m_0, m_1]$, and $|A_\sigma| \leq m_2$ for $\sigma \in [0,1)$.

To prove Theorem 4.1, as an auxiliary problem, we first consider the following equations:

$$\begin{cases} \lambda v - Div(\mu D(v) - \theta I) = 0, \quad div v = 0 & \text{in } \mathbb{R}^N_+, \\ (\mu D(v) - \theta I)n_0 = h & \text{on } \mathbb{R}^N_0, \end{cases} \tag{4.6}$$

and we shall prove the following theorem, which was essentially proved by Shibata et al.[40]

Theorem 4.3 Let $1 < q < \infty$, $\varepsilon \in (0, \pi/2)$, and $\lambda_0 > 0$. Let

$$Y'_q(\mathbb{R}^N_+) = \{(F_3, F_4) \mid F_3 \in L_q(\mathbb{R}^N_+)^N, \quad F_4 \in H^1_q(\mathbb{R}^N)^N\},$$
$$\hat{H}^1_q(\mathbb{R}^N_+) = \{\theta \in L_{q,loc}(\mathbb{R}^N_+) \mid \nabla\theta \in L_q(\mathbb{R}^N_+)\}.$$

Then, there exists a solution operator $V(\lambda) \in Hol(\Sigma_{\varepsilon,\lambda_0}, L(Y'(\mathbb{R}^N_+), H^2_q(\mathbb{R}^N_+)^N))$ such that for any $\lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\lambda_0}$ and $h \in H^1_q(\mathbb{R}^N_+)^N$, $v = V(\lambda)(\lambda^{1/2}h, h)$ are unique solutions of Eq. (4.3) with some $\theta \in \hat{H}^1_q(\mathbb{R}^N_+)$ and

$$R_{L(Y'_q(\mathbb{R}^N), H^{2-j}_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}V(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq r_b(\lambda_0)$$

for $\ell = 0,1$, and $j = 0,1,2$. Here, $r_b(\lambda_0)$ is a constant depending on m_0, m_1, m_2, ε, λ_0, N, and q.

Proof: To prove Theorem 4.3, we start with the solution formulas of Eq. (4.3), which were obatined in Shibata et al.,[40] essentially, but for the sake of the completeness of the paper as much as possible and also for the later use, we will derive them in the following. Applying the partial Fourier transform with respect to $x' = (x_1, \ldots, x_{N-1})$ to Eq. (4.3), we have

$$\begin{cases} \lambda\hat{v}_j + \mu|\xi'|^2 - \partial_N^2\hat{v}_j + i\xi_j\hat{\theta} = 0, \quad \lambda\hat{v}_N + \mu|\xi'|^2 - \partial_N^2\hat{v}_N + \partial_N\hat{\theta} = 0 & (x_N > 0) \\ \sum_{j=1}^{N-1}\xi_j\hat{v}_j + \partial_N\hat{v}_N = 0 & (x_N > 0), \\ \mu(\partial_N\hat{v}_j + i\xi_j\hat{v}_N) = g_j, \quad 2\mu\partial_N\hat{v}_N - \hat{\theta} = g_N & \text{for } x_N = 0. \end{cases} \tag{4.7}$$

Here, for $f = f(x', x_N)$, $x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$, $x_N \in (a,b)$, $\hat{f}$ denotes the partial Fourier transform of f with respect to x' defined by

$$\hat{f}(\xi', x_N) = F'[f(\cdot, x_N)](\xi') = \int_{\mathbb{R}^{N-1}} e^{-ix'\cdot\xi'} f(x', x_N)\,dx'$$

with $\xi' = (\xi_1, \ldots, \xi_{N-1}) \in \mathbb{R}^{N-1}$ and $x' \cdot \xi' = \sum_{j=1}^{N-1} x_j\xi_j$, and we have set $g_j = \hat{h}_j(\xi', 0)$. To obtain solution formula, we set

$$\hat{v}_j = \alpha_j e^{-Ax_N} + \beta_j e^{-Bx_N}, \quad \hat{\theta} = \omega e^{-Ax_N}$$

with $A = |\xi'|$ and $B = \sqrt{\lambda\mu^{-1} + |\xi'|^2}$, and then from (4.7) we have

$$\mu\alpha_j(B^2 - A^2) + i\xi_i\omega = 0, \quad \mu\alpha_N(B^2 - A^2) - A\omega = 0, \tag{4.8}$$

$$\sum_{k=1}^{N-1} i\xi_k\alpha_k - A\alpha_N = 0, \quad \sum_{k=1}^{N-1} i\xi_k\beta_k - B\beta_N = 0, \tag{4.9}$$

$$\mu\{(A\alpha_j + B\beta_j) - i\xi_j(\alpha_N + \beta_N)\} = g_j, \tag{4.10}$$

$$2\mu(A\alpha_N + B\beta_N) + \omega = g_N. \tag{4.11}$$

The solution formula of Eq. (4.3) was given in Shibata et al.,[40] but there is an error in the formula in [ref.40, 4.17] such as[40]

$$\mu\{(A\alpha_j + B\beta_j) + i\xi_j(\alpha_N + \beta_N)\} = \hat{h}_j(\xi',0),$$

which should read

$$\mu\{(A\alpha_j + B\beta_j) - i\xi_j(\alpha_N + \beta_N)\} = \hat{h}_j(\xi',0)$$

as (4.10) above. The formulas obtained in are correct, but we repeat here how to obtain α_j , β_j and ω, because this error confuses readers.

We first drive 2×2 system of equations with respect to α_N and β_N. Multiplying (4.10) with $i\xi_j$, summing up the resultant formulas from $j=1$ through $N-1$ and writing $i\xi'\cdot m' = \sum_{j=1}^{N-1} i\xi_j m_j$ for $m_j \in \{\alpha_j, \beta_j, g_j\}$ give

$$\mu A i\xi'\cdot\alpha' + \mu B i\xi'\cdot\beta' + A^2(\alpha_N + \beta_N) = i\xi'\cdot g'. \tag{4.12}$$

By (4.9),

$$i\xi'\cdot\alpha' = A\alpha_N, \quad i\xi'\cdot\beta' = B\beta_N, \tag{4.13}$$

which, combined with (4.12), leads to

$$2A^2\alpha_N + (A^2 + B^2)\beta_N = \mu^{-1} i\xi'\cdot g'. \tag{4.14}$$

By (4.8),

$$\omega = \frac{\mu(B^2 - A^2)}{A}\alpha_N, \tag{4.15}$$

which, combined with (4.11), leads to

$$(A^2 + B^2)\alpha_N + 2AB\beta_N = \mu^{-1} A g_N. \tag{4.16}$$

Thus, setting

$$L = \begin{pmatrix} A^2 + B^2 & 2A^2 \\ 2AB & A^2 + B^2 \end{pmatrix} \quad \text{(Lopatinskimatrix)},$$

we have

$$L\begin{pmatrix} \beta_N \\ \alpha_N \end{pmatrix} = \begin{pmatrix} \mu^{-1} i\xi'\cdot g' \\ \mu^{-1} A g_N \end{pmatrix}.$$

Since

$$\det L = (A^2 + B^2)^2 - 4A^3B = A^4 - 4A^3B + 2A^2B^2 + B^4 = (B - A)D(A,B)$$

with

$$\mathrm{D}(A,B) = B^3 + AB^2 + 3A^2B^2 - A^3,$$

we have

$$L^{-1} = \frac{1}{(B - A)D(A,B)}\begin{pmatrix} A^2 + B^2 & -2A^2 \\ -2AB & A^2 + B^2 \end{pmatrix}.$$

Thus, we have

$$\begin{aligned} \beta_N &= \frac{1}{\mu(B - A)D(A,B)}((A^2 + B^2)i\xi'\cdot g' - 2A^3 g_N), \\ \alpha_N &= \frac{-1}{\mu(B - A)D(A,B)}(2ABi\xi'\cdot g' - (A^2 + B^2)A g_N)). \end{aligned} \tag{4.17}$$

In particular,

$$\hat{v}_N = \alpha_N e^{-Ax_N} + \beta_N e^{-Bx_N} = \alpha_N(e^{-Ax_N} - e^{-Bx_N}) + (\alpha_N + \beta_N)e^{-Bx_N}.$$

We have

$$\begin{aligned}\alpha_N+\beta_N &= \frac{1}{(B-A)D(A,B)}((A^2+B^2-2AB)i\xi'\cdot g'+((A^2+B^2)A-2A^3)g_N)\\ &=\frac{1}{\mu(B-A)D(A,B)}((B-A)^2 i\xi'\cdot g'+A(B^2-A^2)g_N)\\ &=\frac{1}{\mu(B-A)D(A,B)}((B-A)^2 i\xi'\cdot g'+A(B-A)(A+B)g_N)\\ &=\frac{1}{\mu D(A,B)}((B-A)i\xi'\cdot g'+A(A+B)g_N).\end{aligned} \tag{4.18}$$

Setting

$$\mathcal{M}(x_N)=\frac{e^{-Bx_N}-e^{-Ax_N}}{B-A},$$

we have

$$\hat{v}_N=\frac{A}{\mu D(A,B)}\mathcal{M}(x_N)(2Bi\xi'\cdot g'-(A^2+B^2)g_N)+\frac{e^{-Bx_N}}{\mu D(A,B)}((B-A)i\xi'\cdot g'+A(A+B)g_N). \tag{4.19}$$

By (4.15) and (4.17),

$$\begin{aligned}\omega=\frac{\mu(B^2-A^2)}{A}\alpha_N &=\frac{\mu(B^2-A^2)}{A}\frac{-1}{\mu(B-A)D(A,B)}(2ABi\xi'\cdot g'-(A^2+B^2)Ag_N))\\ &=-\frac{(A+B)}{D(A,B)}(2Bi\xi'\cdot g'-(A^2+B^2)g_N))\end{aligned}$$

and so

$$\hat{\theta}=-\frac{(A+B)e^{-Ax_N}}{D(A,B)}(2Bi\xi'\cdot g'-(A^2+B^2)g_N)). \tag{4.20}$$

By (4.8),

$$\begin{aligned}\alpha_j &=-\frac{i\xi_j}{\mu(B^2-A^2)}\omega=\frac{i\xi_j}{\mu(B^2-A^2)}\frac{A+B}{D(A,B)}(2Bi\xi'\cdot g'-(A^2+B^2)g_N))\\ &=\frac{i\xi_j}{\mu(B-A)D(A,B)}(2Bi\xi'\cdot g'-(A^2+B^2)g_N).\end{aligned} \tag{4.21}$$

By (4.10)

$$\beta_j=\frac{1}{\mu B}g_j+\frac{1}{B}(i\xi_j(\alpha_N+\beta_N)-A\alpha_j).$$

By (4.18) and (4.21)

$$i\xi_j(\alpha_N+\beta_N)-A\alpha_j$$

$$\begin{aligned}&=\frac{i\xi_j}{\mu(B-A)D(A,B)}\{(B-A)^2 i\xi'\cdot g'+A(B-A)(A+B)g_N-A(2Bi\xi'\cdot g'-(A^2+B^2)g_N)\}\\ &=\frac{i\xi_j}{\mu(B-A)D(A,B)}\{(A^2-4AB+B^2)i\xi'\cdot g'+2AB^2g_N)\},\end{aligned}$$

and therefore

$$\beta_j=\frac{1}{\mu B}g_j+\frac{i\xi_j}{\mu(B-A)D(A,B)B}\{(A^2-4AB+B^2)i\xi'\cdot g'+2AB^2g_N)\}. \tag{4.22}$$

Combining (4.21) and (4.22) gives

$$\begin{aligned}\hat{v}_j=\frac{e^{-Bx_N}}{\mu B}g_j &+\frac{i\xi_j e^{-Ax_N}}{\mu(B-A)D(A,B)}\{2Bi\xi'\cdot g'-(A^2+B^2)g_N\}\\ &+\frac{i\xi_j e^{-Bx_N}}{\mu(B-A)D(A,B)B}\{(A^2-4AB+B^2)i\xi'\cdot g'+2AB^2g_N)\}\\ &=\frac{1}{\mu B}g_j+Ii\xi'\cdot g'+IIg_N,\end{aligned}$$

with

$$I = \frac{i\xi_j e^{-Ax_N}}{\mu(B-A)D(A,B)} 2B + \frac{i\xi_j e^{-Bx_N}}{\mu(B-A)D(A,B)B}(A^2 - 4AB + B^2),$$

$$II = -\frac{i\xi_j e^{-Ax_N}}{\mu(B-A)D(A,B)}(A^2+B^2) + \frac{i\xi_j e^{-Bx_N}}{\mu(B-A)D(A,B)} 2AB$$

We proceed as follows:

$$I = \frac{i\xi_j (e^{-Ax_N} - e^{-Bx_N})}{\mu(B-A)D(A,B)} 2B + \frac{i\xi_j e^{-Bx_N}}{\mu(B-A)D(A,B)B}(A^2 - 4AB + 3B^2)$$

$$= -\frac{2i\xi_j B\mathcal{M}\ (x_N)}{\mu D(A,B)} + \frac{i\xi_j (3B-A)e^{-Bx_N}}{\mu D(A,B)B};$$

$$II = -\frac{i\xi_j (e^{-Ax_N} - e^{-Bx_N})}{\mu(B-A)D(A,B)}(A^2+B^2) - \frac{i\xi_j e^{-Bx_N}(A^2 - 2AB + B^2)}{\mu(B-A)D(A,B)}$$

$$= \frac{i\xi_j (A^2+B^2)\mathcal{M}\ (x_N)}{\mu D(A,B)} - \frac{i\xi_j e^{-Bx_N}(B-A)}{\mu D(A,B)}.$$

Therefore, we have

$$\hat{v}_j = \frac{e^{-Bx_N}}{\mu B} g_j - \frac{i\xi_j \mathcal{M}\ (x_N)}{\mu D(A,B)}(2Bi\xi' \cdot g' - (A^2+B^2)g_N) + \frac{i\xi_j e^{-Bx_N}}{\mu D(A,B)B}((3B-A)i\xi' \cdot g' - B(B-A)g_N). \tag{4.23}$$

To define solution operators for Eq. (4.3), we make preparations.

Lemma 4.4: Let $s \varepsilon \mathbb{R}$ and $0 < \varepsilon < \pi/2$. Then, there exists a positive constant c depending on ε, m_1 and m_2 such that

$$c(|\lambda|^{1/2} + A) \leq \mathrm{Re} B \leq |B| \leq (\mu^{-1}|\lambda|)^{1/2} + A, \tag{4.24}$$

$$c(|\lambda|^{1/2} + A)^3 \leq |D(A,B)| \leq 6((\mu^{-1}|\lambda|)^{1/2} + A)^3. \tag{4.25}$$

for any $\lambda \in \Sigma_\varepsilon$ and $\mu \in [m_1, m_2]$.

Proof : The inequality in the left side of (4.24) follows immediately from Lemma 3.2. Notice that

$$D(A,B) = B^3 + 3A^2B + AB^2 - A^3 = B(B^2 + 2A^2) + A(A^2 + \mu^{-1}\lambda) - A^3$$

$$= B(\mu^{-1}\lambda + 4A^2) + \mu^{-1}A\lambda.$$

If we consider the angle of $B(\mu^{-1}\lambda + 4A^2)$ and $-\mu^{-1}A\lambda$, then we see easily that $D(A,B) \neq 0$. Thus, studying the following three cases: $R_1|\lambda|^{1/2} \leq A$, $R_1 A \leq |\lambda|^{1/2}$ and $R_1^{-1}A \leq |\lambda|^{1/2} \leq R_1 A$ for sufficient large $R_1 > 0$, we can prove the inequality in the left side of (4.25). The detailed proof was given in Shibata et al.[41] The independence of the constant c of $\lambda \in \Sigma_\varepsilon$ and $\mu \in [m_0, m_1]$ follows from the homogeneity: $\sqrt{\mu^{-1}(m^2\lambda) + (mA)^2} = m\sqrt{\mu^{-1}\lambda + A^2}$ and $D(mA, mB) = m^3 D(A,B)$ for any $m > 0$ and the compactness of the interval $[m_0, m_1]$.

To introduce the key tool of proving the $\mathcal{R}$ boundedness in the half space, we make a definition.

Definition 4.5: Let V be a domain in C, let $\Xi = V \times (R^{N-1} \setminus \{0\})$, and let $m: \Xi \to C$; $(\lambda, \xi') \mapsto m(\lambda, \xi')$ be C^1 with respect to τ, where $\lambda = \gamma + i\tau \in V$, and C^∞ with respect to $\xi' \in R^{N-1} \setminus \{0\}$.

1) $m(\lambda, \xi')$ is called a multiplier of order s with type 1 on Ξ, if the estimates: $|\partial_{\xi'}^{\kappa'} m(\lambda, \xi')| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{s-|\kappa'|}$, $|\partial_{\xi'}^{\kappa'}(\tau\partial_\tau m(\lambda, \xi'))| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{s-|\kappa'|}$ hold for any multi-index $\kappa \in \mathbb{N}_0^N$ and with some constant $C_{\kappa'}$ depending solely on κ' and V.

2) $m(\lambda, \xi')$ is called a multiplier of order s with type 2 on Ξ, if the estimates: $|\partial_{\xi'}^{\kappa'} m(\lambda, \xi')| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{s} |\xi'|^{-|\kappa'|}$, $|\partial_{\xi'}^{\kappa'}(\tau\partial_\tau m(\lambda, \xi'))| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{s} |\xi'|^{-|\kappa'|}$ hold for any multi-index $\kappa \in \mathbb{N}_0^N$ and $(\lambda, \xi') \in \Xi$ with some constant $C_{\kappa'}$ depending solely on κ' and V.

Let $M_{s,i}(V)$ be the set of all multipliers of order s with type i on Ξ for $i = 1, 2$. For $m \in M_{s,i}(V)$, we set $M(m, V) = \max_{|\kappa'| \leq N} C_{\kappa'}$.

Let $\mathcal{F}_{\xi'}^{-1}$ be the inverse partial Fourier transform defined by

$$\mathcal{F}_{\xi'}^{-1}[f(\xi',x_N)](x') = \frac{1}{(2\pi)^{N-1}}\int_{R^{N-1}} e^{ix'\cdot\xi'} f(\xi',x_N)\,d\xi'.$$

Then, we have the following two lemmas which were proved essentially by Shibata et al.[42] Lemma 5.4 and Lemma 5.6].

Lemma 4.6: Let $0<\varepsilon<\pi/2$, $1<q<\infty$, and $\lambda_0>0$. Given $m\in M_{-2,1}(\Lambda_{\sigma,\lambda_0})$, we define an operator $L(\lambda)$ by

$$[L(\lambda)g](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[m(\lambda,\xi')\lambda^{1/2}e^{-B(x_N+y_N)}\hat{g}(\xi',y_N)](x')\,dy_N.$$

Then, we have

$$\mathcal{R}_{\mathcal{L}(L_q(R_+^N),H_q^{2-j}(R_+^N)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\partial_x^\alpha L_i(\lambda))\mid\lambda\in\Lambda_{\sigma,\lambda_0}\})\le r_b(\lambda_0)$$

for any $\ell=0,1$ and $j=0,1,2$. Where τ denotes the imaginary part of λ, and $r_b(\lambda_0)$ is a constant depending on $M(m,\Lambda_{\sigma,\lambda_0})$ ε, λ_0, N, and q.

Lemma 4.7: Let $0<\varepsilon<\pi/2$, $1<q<\infty$, and $\lambda_0>0$. Given $m\in M_{-2,2}(\Lambda_{\sigma,\lambda_0})$, we define operators $L_i(\lambda)$ $(i=1,\dots,4)$ by

$$[L_1(\lambda)g](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[m(\lambda,\xi')Ae^{-B(x_N+y_N)}\hat{g}(\xi',y_N)](x')\,dy_N,$$

$$[L_2(\lambda)g](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[m(\lambda,\xi')Ae^{-A(x_N+y_N)}\hat{g}(\xi',y_N)](x')\,dy_N,$$

$$[L_3(\lambda)g](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[m(\lambda,\xi')A^2\mathcal{M}(x_N+y_N)\hat{g}(\xi',y_N)](x')\,dy_N,$$

$$[L_4(\lambda)g](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[m(\lambda,\xi')\lambda^{1/2}A\mathcal{M}(x_N+y_N)\hat{g}(\xi',y_N)](x')\,dy_N.$$

Then, we have

$$\mathcal{R}_{\mathcal{L}(L_q(R_+^N),H_q^{2-j}(R_+^N)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\partial_x^\alpha L_i(\lambda))\mid\lambda\in\Lambda_{\sigma,\lambda_0}\})\le r_b(\lambda_0)$$

for $\ell=0,1$ and $j=0,1,2$. Where τ denotes the imaginary part of λ, and $r_b(\lambda_0)$ is a constant depending on $M(m,\Lambda_{\sigma,\lambda_0})$ ε, λ_0, N, and q.

To construct solution operators, we use the following lemma.

Lemma 4.8: Let $0<\varepsilon<\pi/2$, $1<q<\infty$ and $\lambda_0>0$. Given multipliers, $n_1\in M_{-2,1}(\Lambda_{\sigma,\lambda_0})$, $n_2\in M_{-2,2}(\Lambda_{\sigma,\lambda_0})$, and $n_3\in M_{-1,2}(\Lambda_{\sigma,\lambda_0})$, we define operators $T_i(\lambda)$ $(i=1,2,3)$ by

$$T_1(\lambda)h = \mathcal{F}_{\xi'}^{-1}[\lambda^{1/2}e^{-Bx_N}n_1(\lambda,\xi')\hat{h}(\xi',0)](x'),$$

$$T_2(\lambda)h = \mathcal{F}_{\xi'}^{-1}[Ae^{-Bx_N}n_2(\lambda,\xi')\hat{h}(\xi',0)](x'),\mathcal{L}$$

$$T_3(\lambda)h = \mathcal{F}_{\xi'}^{-1}[A\mathcal{M}(x_N)n_3(\lambda,\xi')\hat{h}(\xi',0)](x').$$

Let

$$\mathbb{Z}_q(\mathbb{R}_+^N) = \{(G_1,G_2)\mid G_1\in L_q(\mathbb{R}_+^N),\quad G_2\in H_q^1(\mathbb{R}_+^N)\}.$$

Then, there exist operator families $\mathcal{T}_i(\lambda)\in \mathrm{Hol}(\Lambda_{\sigma,\lambda_0},\mathcal{L}(\mathcal{Z}_q(\mathbb{R}_+^N),H_q^2(\mathbb{R}_+^N)))$ such that for any $\lambda=\gamma+i\tau\in\Lambda_{\sigma,\lambda_0}$ and $h\in H_q^1(\mathbb{R}_+^N)$, $T_i(\lambda)h=\mathcal{T}_i(\lambda)(\lambda^{1/2}h,h)$ and

$$\mathcal{R}_{\mathcal{L}(\mathcal{Z}_q(R_+^N),H_q^{2-j}(R_+^N))}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{T}_i(\lambda))\mid\lambda\in\Lambda_{\sigma,\lambda_0}\})\le r_b(\lambda_0) \qquad (4.26)$$

for $\ell=0,1$, $j=0,1,2$. Where $r_b(\lambda_0)$ is a constant depending on $M(n_i,\Lambda_{\sigma,\lambda_0})$ $(i=1,2,3)$, ε, λ_0, N, and q.

Proof : By Volevich's trick we write

$$\begin{aligned}
T_1(\lambda)h &= -\int_0^\infty \mathcal{F}_{\xi'}^{-1}[\frac{\partial}{\partial y_N}(\lambda^{1/2}e^{-B(x_N+y_N)}n_1(\lambda,\xi')\hat{h}(\xi',y_N))](x')\,dy_N\\
&= -\int_0^\infty \mathcal{F}_{\xi'}^{-1}[\lambda^{1/2}e^{-B(x_N+y_N)}n_1(\lambda,\xi')\partial_N\hat{h}(\xi',y_N)](x')\,dy_N\\
&\quad+\int_0^\infty \mathcal{F}_{\xi'}^{-1}[\lambda^{1/2}e^{-B(x_N+y_N)}\frac{\lambda^{1/2}}{\mu B}n_1(\lambda,\xi')\lambda^{1/2}\hat{h}(\xi',y_N)](x')\,dy_N\\
&\quad-\sum_{j=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}[Ae^{-B(x_N+y_N)}\frac{\lambda^{1/2}}{B}\frac{i\xi_j}{A}n_1(\lambda,\xi')\mathcal{F}[\partial_j h(\cdot,y_N)]](x')\,dy_N,
\end{aligned}$$

where we have used the formula:

$$B = \frac{\mu^{-1}\lambda + A^2}{\mu B} = \frac{\lambda}{\mu B} - \sum_{j=1}^{N-1} \frac{A}{B}\frac{i\xi_j}{A} i\xi_j.$$

Let

$$T_1(\lambda)(G_1, G_2) = -\int_0^\infty \mathcal{F}_{\xi'}^{-1}[\lambda^{1/2} e^{-B(x_N+y_N)} n_1(\lambda,\xi')\mathcal{F}[\partial_N G_2(\cdot, y_N)]](x')dy_N$$

$$+\int_0^\infty \mathcal{F}_{\xi'}^{-1}[\lambda^{1/2} e^{-B(x_N+y_N)} \frac{\lambda^{1/2}}{\mu B} n_1(\lambda,\xi')\mathcal{F}[G_2(\cdot, y_N)]](x')dy_N$$

$$-\sum_{j=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}[A e^{-B(x_N+y_N)} \frac{\lambda^{1/2}}{B}\frac{i\xi_j}{A} n_1(\lambda,\xi')\mathcal{F}[\partial_j G_2(\cdot, y_N)]](x')dy_N,$$

and then, $T_1(\lambda)h = T_1(\lambda)(\lambda^{1/2}h, h)$. Moreover, Lemma 4.6 and Lemma 4.7 yield (4.26) with $j=1$, because

$$n_1(\lambda,\xi') \in M_{-2,1}(\Lambda_{\sigma,\lambda_0}), \quad \frac{\lambda^{1/2}}{\mu B} n_1(\lambda,\xi') \in M_{-2,1}(\Lambda_{\sigma,\lambda_0}), \quad \frac{\lambda^{1/2}}{B}\frac{i\xi_j}{A} n_1(\lambda,\xi') \in M_{-2,2}(\Lambda_{\sigma,\lambda_0}).$$

Analogously, we can prove the existence of $T_2(\lambda)$. To construct $T_3(\lambda)$, we use the formula:

$$\frac{\partial}{\partial x_N}\mathcal{M}(x_N) = -e^{-Bx_N} - A\mathcal{M}(x_N),$$

and then, by Volevich's trick we have

$$T_3(\lambda)h = -\int_0^\infty \mathcal{F}_{\xi'}^{-1}[\frac{\partial}{\partial y_N}(A\mathcal{M}(x_N+y_N) n_3(\lambda,\xi')\hat{h}(\xi', y_N))](x')dy_N = -I + II$$

with

$$I = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[A\mathcal{M}(x_N+y_N) n_3(\lambda,\xi')\partial_N \hat{h}(\xi', y_N)](x')dy_N;$$

$$II = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[(A e^{-B(x_N+y_N)} + A^2\mathcal{M}(x_N+y_N)) n_3(\lambda,\xi')\hat{h}(\xi', y_N)](x')dy_N.$$

Using the formula:

$$1 = \frac{B^2}{B^2} = \frac{\lambda^{1/2}}{\mu B^2}\lambda^{1/2} + \frac{A}{B^2}A = \frac{\lambda^{1/2}}{\mu B^2}\lambda^{1/2} - \sum_{j=1}^{N-1}\frac{i\xi_j}{B^2} i\xi_j,$$

we have

$$I = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[\lambda^{1/2} A\mathcal{M}(x_N+y_N)\frac{\lambda^{1/2}}{\mu B^2} n_3(\lambda,\xi')\partial_N \hat{h}(\xi', y_N)](x')dy_N$$

$$+\int_0^\infty \mathcal{F}_{\xi'}^{-1}[A^2\mathcal{M}(x_N+y_N)\frac{A}{B^2} n_3(\lambda,\xi')\partial_N \hat{h}(\xi', y_N)](x')dy_N;$$

$$II = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[(A e^{-B(x_N+y_N)} + A^2\mathcal{M}(x_N+y_N))\frac{\lambda^{1/2}}{\mu B^2} n_3(\lambda,\xi')\lambda^{1/2}\hat{h}(\xi', y_N)](x')dy_N$$

$$-\sum_{j=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}[(A e^{-B(x_N+y_N)} + A^2\mathcal{M}(x_N+y_N))\frac{i\xi_j}{B^2} n_3(\lambda,\xi')\mathcal{F}[\partial_j h(\cdot, y_N)]](x')dy_N.$$

Let

$$T_3(\lambda)(G_1, G_2)$$

$$= -\int_0^\infty \mathcal{F}_{\xi'}^{-1}[\lambda^{1/2} A\mathcal{M}(x_N+y_N)\frac{\lambda^{1/2}}{\mu B^2} n_3(\lambda,\xi')\mathcal{F}[\partial_N G_2(\cdot, y_N)]](x')dy_N$$

$$-\int_0^\infty \mathcal{F}_{\xi'}^{-1}[A^2\mathcal{M}(x_N+y_N)\frac{A}{B^2} n_3(\lambda,\xi')\mathcal{F}[\partial_N G_2(\cdot, y_N)]](x')dy_N$$

$$+\int_0^\infty \mathcal{F}_{\xi'}^{-1}[(A e^{-B(x_N+y_N)} + A^2\mathcal{M}(x_N+y_N))\frac{\lambda^{1/2}}{\mu B^2} n_3(\lambda,\xi')\mathcal{F}[G_1(\cdot, y_N)]](x')dy_N$$

$$-\sum_{j=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}[(A e^{-B(x_N+y_N)} + A^2\mathcal{M}(x_N+y_N))\frac{i\xi_j}{B^2} n_3(\lambda,\xi')\mathcal{F}[\partial_j G_2(\cdot, y_N)]](x')dy_N,$$

and then $T_3(\lambda)h = T_3(\lambda)(\lambda^{1/2}h, h)$. Moreover, Lemma 4.7 yields (4.26) for $j=3$, because

$$n_3(\lambda)\in M_{-2,2}(\Lambda_{\sigma,\lambda_0}),\quad \frac{\lambda^{1/2}}{\mu B^2}n_3(\lambda,\xi')\in M_{-2,2}(\Lambda_{\sigma,\lambda_0}),\quad \frac{i\xi_j}{B^2}n_3(\lambda,\xi')\in M_{-2,2}(\Lambda_{\sigma,\lambda_0}).$$

This completes the proof of Lemma 4.8.

Continuation of proof of theorem 4.3. Let $v_j(x)=\mathcal{F}_{\xi'}^{-1}[\hat v_j(\xi',x_N)](x')$, and then by (4.19) and (4.23) we have

$$v_N=\mathcal{F}_{\xi'}^{-1}[\frac{A}{\mu D(A,B)}\mathcal{M}(x_N)(2B\sum_{\ell=1}^{N-1}i\xi_\ell\hat h_\ell(\xi',0)-(A^2+B^2)\hat h_N(\xi',0))](x')$$
$$+\mathcal{F}_{\xi'}^{-1}[\frac{Ae^{-Bx_N}}{\mu D(A,B)}((B-A)\sum_{\ell=1}^{N-1}\frac{i\xi_\ell}{A}\hat h_\ell(\xi',0)+(A+B)\hat h_N(\xi',0))](x');$$
$$v_k=\mathcal{F}_{\xi'}^{-1}[\frac{\lambda^{1/2}}{\mu^2B^3}\lambda^{1/2}e^{-Bx_N}\hat h_k(\xi',0)](x')+\mathcal{F}_{\xi'}^{-1}[\frac{A}{\mu B^3}Ae^{-Bx_N}\hat h_k(\xi',0)](x')$$
$$-\mathcal{F}_{\xi'}^{-1}[A\mathcal{M}(x_N)\frac{i\xi_k}{A}\frac{1}{\mu D(A,B)}(2B\sum_{\ell=1}^{N-1}i\xi_\ell\hat h_\ell(\xi',0)-(A^2+B^2)\hat h_N(\xi',0))](x')$$
$$+\mathcal{F}_{\xi'}^{-1}[Ae^{-Bx_N}\frac{i\xi_k}{A}\frac{1}{\mu D(A,B)B}((3B-A)\sum_{\ell=1}^{N-1}i\xi_\ell\hat h_\ell(\xi',0)-B(B-A)\hat h_N(\xi',0))](x'),$$

for $k=1,\dots,N-1$, where we have used the formula $\frac{1}{\mu B}=\frac{\lambda}{\mu^2B^3}+\frac{A^2}{\mu B^3}$ to treat the first term of $\hat v_j$ in (4.23). Since

$$\frac{Bi\xi_\ell}{\mu D(A,B)},\frac{A^2+B^2}{\mu D(A,B)},\frac{i\xi_k}{A}\frac{Bi\xi_\ell}{\mu D(A,B)},\frac{i\xi_k}{A}\frac{A^2+B^2}{\mu D(A,B)}\in M_{-1,2}(\Sigma_{\varepsilon,\lambda_0}),$$
$$\frac{B-A}{\mu D(A,B)}\frac{i\xi_\ell}{A},\frac{A+B}{\mu D(A,B)},\frac{A}{\mu B^3},\frac{i\xi_k}{A}\frac{(3B-A)i\xi_\ell}{\mu D(A,B)B},\frac{i\xi_k}{A}\frac{B(B-A)}{\mu D(A,B)B}\in M_{-2,2}(\Sigma_{\varepsilon,\lambda_0}),$$

and $\frac{\lambda^{1/2}}{\mu^2B^3}\in M_{-2,1}(\Sigma_{\varepsilon,\lambda_0})$, by Lemma 4.8 we have Theorem 4.3. We next consider the equations:

$$\begin{cases}\lambda w-\mathrm{Div}(\mu D(w)-qI)=0,\quad \mathrm{div} w &=0 & \text{in } \mathbb{R}^N_+,\\ \lambda h+A_\sigma\cdot\nabla'h-w\cdot n_0 &=d & \text{on } \mathbb{R}^N_0,\\ (\mu D(w)-qI)n_0-\delta(\Delta'h)n_0 &=0 & \text{on } \mathbb{R}^N_0.\end{cases}\tag{4.27}$$

We shall prove the following theorem.

Theorem 4.9: Let $1<q<\infty$ and $\varepsilon\in(0,\pi/2)$. Then, there exist a $\lambda_1>0$ and solution operators $\mathcal{W}(\lambda)$ and $\mathcal{H}_\sigma(\lambda)$ with

$$\mathcal{W}(\lambda)\in \mathrm{Hol}(\Lambda_{\sigma,\lambda_1},\mathcal{L}(H_q^2(\mathbb{R}^N_+),H_q^2(\mathbb{R}^N_+)^N)),\quad \mathcal{H}_\sigma(\lambda)\in \mathrm{Hol}(\Lambda_{\sigma,\lambda_1},\mathcal{L}(H_q^2(\mathbb{R}^N_+),H_q^3(\mathbb{R}^N_+))),$$

such that for any $\lambda=\gamma+i\tau\in\Lambda_{\sigma,\lambda_1}$ and $d\in H_q^2(\mathbb{R}^N_+)$, $w=\mathcal{W}(\lambda)d$ and $h=\mathcal{H}_\sigma(\lambda)d$ are unique solutions of Eq. (4.27) with some $q\in\hat H_q^1(\Omega)$, and

$$\mathcal{R}_{\mathcal{L}(H_q^2(\mathbb{R}^N_+),H_q^{2-k}(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{W}(\lambda))\mid\lambda\in\Lambda_{\sigma,\lambda_1}\})\le r_b(\lambda_1),$$
$$\mathcal{R}_{\mathcal{L}(H_q^2(\mathbb{R}^N_+),H_q^{3-m}(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^\ell(\lambda^m\mathcal{H}_\sigma(\lambda))\mid\lambda\in\Lambda_{\sigma,\lambda_1}\})\le r_b(\lambda_1)$$

for $\ell=0,1$, $k=0,1,2$, and $m=0,1$, where $r_b(\lambda_1)$ is a constant depending on m_0, m_1, m_2, ε, λ_1, N, and q.

Proof: We start with solution formulas. Applying the partial Fourier transform to Eq. (4.27), we have the following generalized resolvent problem:

$$\lambda\hat w_j+\mu|\xi'|^2\hat w_j-\mu\partial_N^2\hat w_j+i\xi_jq=0\quad(x_N>0),$$
$$\lambda\hat w_N+\mu|\xi'|^2\hat w_N-\mu\partial_N^2\hat w_N+\partial_Nq=0\quad(x_N>0),$$
$$\sum_{j=1}^{N-1}i\xi_j\hat w_j+\partial_N\hat w_N=0\quad(x_N>0),$$
$$\mu(\partial_N\hat w_j(0)+i\xi_j\hat w_N(0))=0,\quad 2\mu\partial_N\hat w_N-q=\sigma A^2\hat h\quad \text{for } x_N=0,$$
$$\lambda\hat h+\sum_{j=1}^{N-1}i\xi_jA_{\sigma j}\hat h+\hat w_N=\hat d\quad \text{for } x_N=0.\tag{4.28}$$

Where, we have set $A_\sigma = (A_{\sigma 1}, \dots, A_{\sigma N-1})$. Using the solution formulas given in (4.19) and (4.23) with $g_j = 0 \quad (j = 1, \dots, N-1)$, and $g_N = \sigma A^2 \hat{h}$, we have

$$
\begin{aligned}
\hat{w}_j &= \frac{i\xi_j \mathcal{M}(x_N)}{\mu D(A,B)} \sigma A^2 (A^2 + B^2)\hat{h} - \frac{i\xi_j e^{-Bx_N}}{\mu D(A,B)} \sigma A^2 (B - A)\hat{h}, \\
\hat{w}_N &= -\frac{A\mathcal{M}(x_N)}{\mu D(A,B)} \sigma A^2 (A^2 + B^2)\hat{h} + \frac{e^{-Bx_N}}{\mu D(A,B)} \sigma A^3 (A + B)\hat{h}, \\
q &= -\frac{(A+B)A^2(A^2+B^2)e^{-Ax_N}}{D(A,B)} \hat{h}
\end{aligned} \tag{4.29}
$$

Inserting the formula of $\hat{w}_N |_{x_N=0}$ into the last equation in (4.28), we have

$$(\lambda + i\xi' \cdot A_\sigma)\hat{h} + \frac{\sigma A^3 (A+B)}{\mu D(A,B)} \hat{h} = \hat{d},$$

where we have set $i\xi' \cdot A_\sigma = \sum_{j=1}^{N-1} i\xi_j A_{\sigma j}$, which implies that

$$\hat{h} = \frac{\mu D(A,B)}{E_\sigma} \hat{d} \tag{4.30}$$

with $E_\sigma = \mu(\lambda + i\xi' \cdot A_\sigma) D(A,B) + \sigma A^3 (A+B)$. Thus, we have the following solution formulas:

$$
\begin{aligned}
\hat{w}_j &= i\xi_j \mathcal{M}(x_N) \frac{\sigma A^2 (A^2 + B^2)}{E_\sigma} \hat{d} - i\xi_j e^{-Bx_N} \frac{\sigma A^2 (B - A)}{E_\sigma} \hat{d}, \\
\hat{w}_N &= -A\mathcal{M}(x_N) \frac{\sigma A^2 (A^2 + B^2)}{E_\sigma} \hat{d} + e^{-Bx_N} \frac{\sigma A^3 (A + B)}{E_\sigma} \hat{d}, \\
\hat{q} &= -\frac{\mu (A+B)A^2(A^2+B^2)e^{-Ax_N}}{E_\sigma} \hat{d}.
\end{aligned} \tag{4.31}
$$

Concerning the estimation for E_σ, we have the following lemma.

Lemma 4.10:

(1) Let $0 < \varepsilon < \pi/2$ and let E_0 be the function defined in (4.30) with $A_0 = 0$. Then, there exists a $\lambda_1 > 0$ and $c > 0$ such that the estimate:

$$|E_0| \geq c(|\lambda| + A)(|\lambda|^{1/2} + A)^3 \tag{4.32}$$

holds for $(\lambda, \xi') \in \Sigma_{\varepsilon, \lambda_1} \times (\mathbb{R}^{N-1} \setminus \{0\})$.

(2) Let $\sigma \in (0,1)$ and let E_σ be the function defined in (4.30). Then, there exists a $\lambda_1 > 0$ and $c > 0$ such that

$$|E_\sigma| \geq c(|\lambda| + A)(|\lambda|^{1/2} + A)^3 \tag{4.33}$$

holds for $(\lambda, \xi') \in C_{+, \lambda_1} \times (\mathbb{R}^{N-1} \setminus \{0\})$.

Where, the constant c in 1 and 2 depends on λ_1, m_0, m_1, and m_2.

Proof : We first study the case where $|\lambda| \geq R_1 A$ for large $R_1 > 0$. Since $|B| \leq A + \mu^{-1/2} |\lambda|^{1/2}$ and since $\Lambda_{\sigma, \lambda_1} \subset \Sigma_\varepsilon$, by Lemma 4.4 we have

$$
\begin{aligned}
|E_\sigma| &\geq \mu |\lambda| \| D(A,B) | - \mu |A_\sigma| \| A \| D(A,B) | - \sigma A^3 (A + \mu^{-1} |\lambda|^{1/2}) \\
&\geq c\mu |\lambda| (|\lambda|^{1/2} + A)^3 - \mu m_2 C R_1^{-1} |\lambda| (|\lambda|^{1/2} + A)^3 - \mu^{-1/2} \sigma |\lambda|^{1/2} (|\lambda|^{1/2} + A)^3 \\
&\geq (c\mu/2) |\lambda| (|\lambda|^{1/2} + A)^3 + ((c\mu/2) - \mu m_2 C R_1^{-1} - \sigma / (\mu |\lambda|)^{1/2}) |\lambda| (|\lambda|^{1/2} + A)^3.
\end{aligned}
$$

Thus, choosing $R_1 > 0$ and $\lambda_1 > 0$ so large that $(c\mu/4) - \mu m_2 C R_1^{-1} \geq 0$ and $(c\mu/4) - \sigma / (\mu \lambda_1)^{1/2} \geq 0$, we have

$$|\tilde{E}_\sigma| \geq (c\mu/2) |\lambda| (|\lambda|^{1/2} + A)^3 \geq (c\mu/4)(|\lambda| + R_1 A)(|\lambda|^{1/2} + A)^3 \tag{4.34}$$

provided that $|\lambda| \geq R_1 A$ and $\lambda \in \Lambda_{\sigma, \lambda_1}$. When $\sigma = 0$, we may assume that $m_2 = 0$ above.

We now consider the case where $|\lambda| \leq R_1 A$. We first consider the case of $\sigma = 0$. We assume that $\lambda \in \Sigma_{\varepsilon, \lambda_1}$. In this case, we have $A \geq R_1^{-1} |\lambda|^{1/2} \lambda_1^{1/2}$, and so, setting $R_2 = R_1^{-1} \lambda_1^{1/2}$ and choosing R_2 large enough, we have $B = A(1 + O(R_2^{-1}))$. In particular, $D(A,B) = 4A^3 (1 + O(R_2^{-1}))$. Thus, we have

$$E_0 = 4\mu\lambda A^3 (1 + O(R_2^{-1})) + 2\sigma A^4 (1 + O(R_2^{-1})).$$

Using Lemma 3.2, we have

$$|E_0| \geq |4\mu\lambda A^3 + 2\sigma A^4| - 4\mu|\lambda|A^3 O(R_2^{-1}) - 2\sigma A^4 O(R_2^{-1})$$
$$\geq (\sin\varepsilon)(4\mu|\lambda|A^3 + 2\sigma A^4) - O(R_2^{-1})(4\mu|\lambda|A^3 + 2\sigma A^4).$$

Thus, choosing $R_2 > 0$ so large that $(\sin\varepsilon/2) - O(R_2^{-1}) \geq 0$, we have

$$|E_0| \geq (\sin\varepsilon/2)(4\mu|\lambda|A^3 + 2\sigma A^4) \geq c(|\lambda + A)A^3 \geq (c/2)(|\lambda| + A)(A + R_2|\lambda|^{1/2})^3.$$

This completes the proof of 1.

We next consider the case of $\sigma \in (0,1)$. We assume that $|\lambda| \leq R_1 A$ and $\lambda \in C_{+,\lambda_1}$. In this case, we have $A \geq R_1^{-1}|\lambda|^{1/2}\lambda_1^{1/2}$, and so, setting $R_2 = R_1^{-1}\lambda_1^{1/2}$ and choosing R_2 large enough, we have $B = A(1 + O(R_2^{-1}))$. In particular, $D(A,B) = 4A^3(1 + O(R_2^{-1}))$. Thus, we have

$$E_\sigma = 4\mu(\mathrm{Re}\lambda + i(\mathrm{Im}\lambda + A_\sigma \cdot \xi'))A^3(1 + O(R_2^{-1})) + 2\sigma A^4(1 + O(R_2^{-1})),$$

and so, taking the real part gives

$$\mathrm{Re}E_\sigma = 4\mu(\mathrm{Re}\lambda)A^3(1 + O(R_2^{-1})) + O(R_2^{-1})(\mathrm{Im}\lambda + A_\sigma \cdot \xi')A^3 + 2\sigma A^4(1 + O(R_2^{-1})).$$

Since $\mathrm{Re}\lambda \geq \lambda_1 > 0$ and $|\lambda| \leq R_1 A$, we have

$$\mathrm{Re}E_\sigma \geq 2\sigma A^4 - (4\mu(m_2 + R_1) + 2\sigma)O(R_2^{-1})A^4,$$

and so, choosing $R_2 > 0$ so large that $\sigma - (4\mu(m_2 + R_1) + 2\sigma)O(R_2^{-1}) \geq 0$, we have

$$|E_\sigma| \geq \mathrm{Re}E_\sigma \geq \sigma A^4 \geq (\sigma/2^4)(A + R_1^{-1}|\lambda|)(A + R_2|\lambda|^{1/2})^3.$$

This completes the proof of Lemma 4.10.

Continuation of proof of theorem 4.9: Let $w_j = \mathcal{F}_{\xi'}^{-1}[\hat{w}_j]$, $q = \mathcal{F}_{\xi'}^{-1}[q]$ and $\eta = \varphi(x_N)\mathcal{F}_{\xi'}^{-1}[e^{-Ax_N}\hat{h}]$, where $\varphi \in C_0^\infty(\mathbb{R})$ equals to 1 for $x_N \in (-1,1)$ and 0 for $x_N \notin [-2,2]$. Notice that $\eta|_{x_N=0} = h$.

Let $w_j(x) = \mathcal{F}_{\xi'}^{-1}[\hat{w}_j(\xi', x_N)](x')$. In view of (4.31) and Volevich's trick, we define $\mathcal{W}_j(\lambda)$ by

$$\mathcal{W}_j(\lambda)d = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[-(Ae^{-B(x_N+y_N)} + A^2\mathcal{M}\ (x_N + y_N))\frac{i\xi_j}{A}\frac{\sigma(A^2+B^2)}{E_\sigma}\mathcal{F}[\Delta' d](\xi', y_N)$$
$$+ Ae^{-B(x_N+y_N)}\frac{i\xi_j}{A}\frac{\sigma B(B-A)}{E_\sigma}\mathcal{F}'[\Delta' d](\xi', y_N)](x')dy_N$$
$$+\int_0^\infty \mathcal{F}_{\xi'}^{-1}[-A^2\mathcal{M}\ (x_N + y_N)\frac{\sigma(A^2+B^2)}{E_\sigma}\mathcal{F}'[\partial_j\partial_N d](\xi', y_N)$$
$$+ Ae^{-B(x_N+y_N)}\frac{\sigma A(B-A)}{E_\sigma}\mathcal{F}'[\partial_j\partial_N d](\xi', y_N)](x')dy_N,$$

where we have used $\mathcal{F}'[\Delta' d](\xi', y_N) = -A^2\hat{d}(\xi', y_N)$. We have $\mathcal{W}_j(\lambda)d = w_j$. By Lemma 4.10, we see that

$$\frac{A^2+B^2}{E_\sigma}, \quad \frac{A^2+B^2}{E_\sigma}\frac{\xi_j}{A}, \quad \frac{A(B-A)}{E_\sigma}, \quad \frac{B(B-A)}{E_\sigma}\frac{\xi_j}{A}$$

belong to $M_{-2,2}(\Lambda_{\sigma,\lambda_1})$, and so by Lemma 4.7, we have

$$\mathcal{R}_{\mathcal{L}(H_q^2(R_+^N), H_q^{2-k}(R_+^N))}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2}\mathcal{W}_j(\lambda)) \mid \lambda \in \Lambda_{\sigma,\lambda_1}\}) \leq r_b(\lambda_1)$$

for $\ell = 0,1$ and $k = 0,1,2$, where $r_b(\lambda_1)$ is a constant depending on m_0, m_1, m_2 and λ_1.

Analogously, $\mathcal{W}_N(\lambda)$ can be constructed. Thus, our final task is to construct $\mathcal{H}_\sigma(\lambda)$. In view of (4.30), we define $\mathcal{H}_\sigma(\lambda)$ acting on $d \in H_q^2(R_+^N)$ by

$$\mathcal{H}_\sigma(\lambda)d = \phi(x_N)\mathcal{F}_{\xi'}^{-1}[e^{-Ax_N}\frac{\mu D(A,B)}{E_\sigma}\hat{d}(\xi', 0)](x').$$

Since $\varphi(x_N)$ equals one for $x_N \in (-1,1)$, we have $\mathcal{H}_\sigma(\lambda)d|_{x_N=0} = h$. Recalling the definition of $\hat{h}$ given in (4.30) and using Volevich's trick, we have $\mathcal{H}_\sigma(\lambda)d = \phi(x_N)\{\Omega_\sigma(\lambda)d + \mathcal{H}_\sigma^2(\lambda)d\}$ with

$$\Omega_\sigma(\lambda)d = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[Ae^{-A(x_N+y_N)}\frac{\mu D(A,B)}{E_\sigma}\varphi(y_N)\hat{d}(\xi', y_N)](x')dy_N,$$

$$H_\sigma^2(\lambda)d = -\int_0^\infty \mathcal{F}_{\xi'}^{-1}[e^{-A(x_N+y_N)} \frac{\mu D(A,B)}{E_\sigma} \partial_N(\varphi(y_N)\hat{d}(\xi',y_N))](x')dy_N.$$

We use the following lemma.

Lemma 4.11: Let Λ be a domain in C and let $1<q<\infty$. Let φ and ψ be two $C_0^\infty((-2,2))$ functions. Given $m \in M_{0,2}(\Lambda)$, we define operators $L_6(\lambda)$ and $L_7(\lambda)$ acting on $g \in L_q(\mathbb{R}_+^N)$ by

$$[L_6(\lambda)g](x) = \varphi(x_N)\int_0^\infty \mathcal{F}_{\xi'}^{-1}[e^{-A(x_N+y_N)} m(\lambda,\xi')\hat{g}(\xi',y_N)\psi(y_N)]dy_N,$$

$$(x) = \varphi(x_N)\int_0^\infty \mathcal{F}_{\xi'}^{-1}[Ae^{-A(x_N+y_N)} m(\lambda,\xi')\hat{g}(\xi',y_N)\psi(y_N)]dy_N.$$

Then,

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}_+^N))}(\{(\tau\partial_\tau)^\ell L_k(\lambda) \mid \lambda \in \Lambda\}) \le r_b \tag{4.35}$$

for $\ell = 0,1$ and $k = 6,7$, where r_b is a constant depending on $M(m,\Lambda)$. Here, $M(m,\Lambda)$ is the number defined in definition 4.5.

Proof : Using Lemma 5.4 in Shibata et al.,[42] we can show (4.35) immediately for $k=7$, and so we show (4.35) only in the case that $k=6$ below. In view of Definition 1.2, for any $n \in N$, we take $\{\lambda_j\}_{j=1}^n \subset \Lambda$, $\{g_j\}_{j=1}^n \subset L_q(\mathbb{R}_+^N)$, and $r_j(u)(j=1,\ldots,n)$ are Rademacher functions. For the notational simplicity, we set

$$|||\, L_6(\lambda)g \,||| = \left\| \sum_{j=1}^n r_j(u)L_6(\lambda_j)g_j \right\|_{L_q((0,1),L_q(\mathbb{R}^N))} = (\int_0^1 \left\| \sum_{j=1}^n r_j(u)L_6(\lambda_j)g_j \right\|_{L_q(\mathbb{R}_+^N)}^q du)^{1/q}.$$

By the Fubini-Tonelli theorem, we have

$$\begin{aligned} |||\, L_6(\lambda)g \,|||^q &= \int_0^1\int_0^\infty\int_{\mathbb{R}^{N-1}} \Big|\sum_{j=1}^n r_j(u)L_6(\lambda_j)g_j\Big|^q dy'dx_N du \\ &= \int_0^\infty (\int_0^1 \left\| \sum_{j=1}^n r_j(u)L_6(\lambda_j)g_j \right\|_{L_q(\mathbb{R}^{N-1})}^q du)dx_N. \end{aligned}$$

Since

$$|\partial_{\xi'}^{\alpha'}(e^{-A(x_N+y_N)}m_0(\lambda,\xi'))| \le C_{\alpha'}|\xi'|^{-|\alpha'|}$$

for any $x_N \ge 0$, $y_N \ge 0$, $(\lambda,\xi') \in \Lambda\times(\mathbb{R}^{N-1}\setminus\{0\})$, and $\alpha' \in \mathbb{R}^{N-1}$, by Theorem 3.1 we have

$$\begin{aligned} &\int_0^1 \left\| \sum_{j=1}^n r_j(u)\mathcal{F}_{\xi'}^{-1}[e^{-A(x_N+y_N)}m(\lambda_j,\xi')\hat{g}_j(\xi',y_N)](y') \right\|_{L_q(\mathbb{R}^{N-1})}^q du \\ &\le C_{N,q}M(m,\Lambda)\int_0^1 \left\| \sum_{j=1}^n r_j(u)g_j(\cdot,y_N) \right\|_{L_q(\mathbb{R}^{N-1})}^q du. \end{aligned} \tag{4.36}$$

For any $x_N \ge 0$, by Minkowski's integral inequality, Lemma 3.3, and Hölder's inequality, we have

$$\begin{aligned} &\left(\int_0^1 \left\| \sum_{j=1}^n r_j(u)L_6(\lambda_j)g_j \right\|_{L_q(\mathbb{R}^{N-1})}^q du \right)^{1/q} \\ &= |\phi(x_N)| \left(\int_0^1 \left\| \int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[\sum_{j=1}^n r_j(u)e^{-A(x_N+y_N)}m(\lambda_j,\xi')\hat{g}_j(\xi',y_N) \right](y')\psi(y_N)dy_N \right\|_{L_q(\mathbb{R}^{N-1})}^q du) \right)^{1/q} \\ &\le |\varphi(x_N)| \int_0^1\int_0^\infty \left\| \mathcal{F}_{\xi'}^{-1}[\sum_{j=1}^n r_j(u)e^{-A(x_N+y_N)}m(\lambda_j,\xi')\hat{g}_j(\xi',y_N)](y')\psi(y_N) \right\|_{L_q(\mathbb{R}^{N-1})} dy_N)^q du)^{1/q} \\ &\le |\varphi(x_N)| \int_0^\infty (\int_0^1 \left\| \mathcal{F}_{\xi'}^{-1}[\sum_{j=1}^n r_j(u)e^{-A(x_N+y_N)}m(\lambda_j,\xi')\hat{g}_j(\xi',y_N)](y') \right\|_{L_q(\mathbb{R}^{N-1})}^q du)^{1/q} |\psi(y_N)| dy_N \end{aligned}$$

$$\le C_{N,q} M(m,\Lambda)|\varphi(x_N)|\int_0^\infty \Big(\int_0^1 \Big\|\sum_{j=1}^n r_j(u) g_j(\cdot,y_N)\Big\|^q_{L_q(\mathbb{R}^{N-1})} du\Big)^{1/q} |\psi(y_N)| dy_N$$

$$\le C_{N,q} M(m,\Lambda)|\varphi(x_N)|\Big(\int_0^\infty\int_0^1 \Big\|\sum_{j=1}^n r_j(u) g_j(\cdot,y_N)\Big\|^q_{L_q(\mathbb{R}^{N-1})} du\,dy_N\Big)^{1/q} \Big(\int_0^\infty |\psi(y_N)|^{q'} dy_N\Big)^{1/q'}$$

$$= C_{n,q} M(m,\Lambda)|\phi(x_N)|\Big(\int_0^1 \Big\|\sum_{j=1}^n r_j(u) g_j(\cdot,y_N)\Big\|^q_{L_q(\mathbb{R}^N_+)} du\Big)^{1/q} \Big(\int_0^\infty |\psi(y_N)|^{q'} dy_N\Big)^{1/q'}$$

Putting these inequalities together and using Hölder's inequality gives

$$\int_0^1 \Big\|\sum_{j=1}^n r_j(u) L_6(\lambda_j) g_j\Big\|^q_{L_q(\mathbb{R}^N_+)} du$$

$$\le (C_{n,q} M(m,\Lambda))^q \int_0^\infty |\varphi(x_N)|^q\, dx_N \int_0^1 \Big\|\sum_{j=1}^n r_j(u) g_j\Big\|^q_{L_q(\mathbb{R}^N_+)} du \Big(\int_0^\infty |\psi(y_N)|^{q'} dy_N\Big)^{q/q'},$$

and so, we have

$$\Big\|\sum_{j=1}^n r_j L_6(\lambda_j) g_j\Big\|_{L_q((0,1),L_q(\mathbb{R}^N_+))} \le C_{n,q} M(m,\Lambda)\|\varphi\|_{L_q(\mathbb{R})}\|\psi\|_{L_{q'}(R)}\Big\|\sum_{j=1}^n r_j g_j\Big\|_{L_q((0,1),L_q(\mathbb{R}^N_+))}.$$

This shows Lemma 4.11.

Continuation of proof of theorem 4.9: For $(j,\alpha',k)\in N_0\times N_0^{N-1}\times N_0$ with $j+|\alpha'|+k\le 3$ and $j=0,1$, we write

$$\lambda^j\partial_{x'}^{\alpha'}\partial_N^k \mathcal{H}_\sigma(\lambda)d = \sum_{n=0k}^{k} C_n(\partial_N^{k-n}\varphi(x_N))[\lambda^j\partial_{x'}^{\alpha'}\partial_N^n\Omega_\sigma(\lambda)d + \lambda^j\partial_{x'}^{\alpha'}\partial_N^n \mathcal{H}_\sigma^2(\lambda)d],$$

and then

$$\lambda^j\partial_{x'}^{\alpha'}\partial_N^n\Omega_\sigma(\lambda)d$$

$$= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[Ae^{A(x_N+y_N)}\frac{\mu\lambda^j(i\xi')^{\alpha'}(-A)^n D(A,B)}{(1+A^2)E_\sigma}\varphi(y_N)\mathcal{F}'[(1-\Delta')d](\xi',y_N)\Big](x')dy_N;$$

$$\lambda^j\mathcal{H}_\sigma^2(\lambda)d = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[e^{-A(x_N+y_N)}\frac{\mu\lambda^j D(A,B)}{E_\sigma}\partial_N(\varphi(y_N)\hat d(\xi',y_N))\Big](x')dy_N;$$

$$\lambda^j\partial_{x'}^{\alpha'}\partial_N^n \mathcal{H}_\sigma^2(\lambda)d$$

$$= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[e^{A(x_N+y_N)}\frac{\mu\lambda^j(i\xi')^{\alpha'}(-A)^n D(A,B)}{(1+A^2)E_\sigma}\partial_N(\varphi(y_N)\hat d(\xi',y_N))\Big](x')dy_N$$

$$-\sum_{k=1}^{N-1}\int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[Ae^{A(x_N+y_N)}\frac{\mu\lambda^j(i\xi')^{\alpha'}(-A)^n D(A,B)}{(1+A^2)E_\sigma}\frac{i\xi_j}{A}\partial_N(\phi(y_N)\mathcal{F}[\partial_j d(\cdot,y_N)](\xi')\}\Big](x')dy_N$$

for $|\alpha'|+n\ge 1$. Where, we have used the formula:

$$1 = \frac{1+A^2}{1+A^2} = \frac{1}{1+A^2} - \sum_{j=1}^{N-1}\frac{A}{1+A^2}\frac{i\xi_j}{A}i\xi_j$$

in the third equality. By Lemma 4.4 and Lemma 4.10, we see that multipliers:

$$\frac{\lambda^j(i\xi')^{\alpha'}A^n D(A,B)}{(1+A^2)E_\sigma},\ \frac{\lambda^j D(A,B)}{E_\sigma},\ \frac{\lambda^j(i\xi')^{\alpha'}A^n D(A,B)}{(1+A^2)E_\sigma},\ \frac{\lambda^j(i\xi')^{\alpha'}A^n D(A,B)}{(1+A^2)E_\sigma}\frac{\xi_j}{A}$$

belong to $M_{0,2}(\Lambda_{\sigma,\lambda_1})$, because $j+|\alpha'|+n\le 3$ and $j=0,1$. Thus, using Lemma 4.7, we see that for any $n\in\mathbb{N}$, $\{\lambda_j\}_{j=1}^n\subset\Lambda_{\sigma,\lambda_1}$, and $\{d_j\}_{j=1}^n\subset H_q^2(\mathbb{R}^N_+)$, the inequality:

$$\Big\|\sum_{\ell=1}^n r_\ell(\cdot)(\partial_N^{k-n}\phi)(\lambda_\ell)^j\partial_{x'}^{\alpha'}\partial_N^n \mathcal{H}_\sigma^i(\lambda_\ell)d_\ell\Big\|_{L_q((0,1),L_q(\mathbb{R}^N))} \le C\Big\|\sum_{\ell=1}^n r_\ell(\cdot)d_\ell\Big\|_{L_q((0,1),H_q^2(\mathbb{R}^N_+))}$$

holds for $i=1,2$, which leads to

$$\Big\|\sum_{\ell=1}^n r_\ell(\cdot)(\lambda_\ell)^j\partial_{x'}^{\alpha'}\partial_N^k \mathcal{H}_\sigma(\lambda_\ell)d_\ell\Big\|_{L_q((0,1),L_q(\mathbb{R}^N_+))} \le C\Big\|\sum_{\ell=1}^n r_\ell(u)d_\ell\Big\|_{L_q((0,1),H_q^2(\mathbb{R}^N_+))}.$$

Here, C is a constant depending on N, q, m_0, m_1, and m_2. This shows that

$$R_{L(H_q^2(R_+^N),H_q^{3-k}(R_+^N))}(\{\lambda^k H_\sigma(\lambda) \mid \lambda\in\Lambda_{\sigma,\lambda_1}\})\le r_b$$

for $k=0,1$. Here, $r_b(\lambda_1)$ is a constant depending on N, q, m_0, m_1, and m_2, but independent of $\mu,\delta\in[m_0,m_1]$ and $|A_\sigma|\le m_2$ for $\sigma\in[0,1)$. Analogously, we have

$$R_{L(H_q^2(R_+^N),H_q^{3-k}(R_+^N))}(\{\tau\partial_\tau(\lambda^k H_\sigma(\lambda)) \mid \lambda\in\Lambda_{\sigma,\lambda_1}\})\le r_b(\lambda_1)$$

for $k=0,1$. This completes the proof of Theorem 4.9.

Proof of theorem 4.1: Let $(f,d,h)\in Y_q(\mathbb{R}_+^N)$. Let $g\in H_q^1(\mathbb{R}_+^N)$ be a solution of the variational equation:

$$\lambda(g,\varphi)_{\mathbb{R}_+^N}+(\nabla g,\nabla\varphi)_{\mathbb{R}_+^N}=(-f,\nabla\phi)_{\mathbb{R}_+^N}\quad\text{for any}\varphi\in \mathrm{H}^1_{q',0}(\mathbb{R}^N_+),\tag{4.37}$$

subject to $g=\rho$ on Γ. Let u, q and h be solutions of the equations:

$$\begin{cases}\lambda u-\mathrm{Div}(\mu D(u)-q\mathrm{I})=f,\quad \mathrm{div}u & =g=\mathrm{div}g & \text{in } \mathbb{R}^N_+,\\ \lambda h+A_\sigma\cdot\nabla_\Gamma h-u\cdot n_0 & =d & \text{on } \mathbb{R}^N_0,\\ (\mu D(u)-qI-\delta(\Delta' h)I)n_0 & =h & \text{on } \mathbb{R}^N_0.\end{cases}\tag{4.38}$$

Where, g is a solution of Eq. (4.37) with $\rho=h\cdot n_0$ and $g=\lambda^{-1}(f+\nabla g)$. Then, according to what pointed out in Subsec. 2.1, u and h are solutions of Eq. (4.3). Thus, we shall look for u, q and h below.

We first consider the equation:

$$\mathrm{d}iv\,\mathrm{v}=g\quad\text{in } \mathbb{R}^N_+.\tag{4.39}$$

We have the following lemma.

Lemma 4.12: Let $1<q<\infty$, $0<\varepsilon<\pi/2$, and $\lambda_0>0$. Let

$$Y_q''(\mathbb{R}_+^N)=\{(f,\rho)\mid f\in L_q(\mathbb{R}_+^N)^N,\rho\in H_q^1(\mathbb{R}_+^N)\},$$

$$Y''_q(\mathbb{R}_+^N)=\{(F_1,G_1,G_2)\mid F_1\in L_q(\mathbb{R}_+^N)^N,G_1\in L_q(\mathbb{R}_+^N),G_2\in H_q^1(\mathbb{R}_+^N)\}.$$

Let g be a solution of the variational problem (4.37). Then, there exists an operator family $B_0(\lambda)\in \mathrm{Hol}(\Sigma_{\varepsilon,\lambda_0},L(Y''_q(\mathbb{R}_+^N),H_q^2(\mathbb{R}_+^N)^N))$ such that for any $\lambda\in\Sigma$ and $(f,\rho)\in Y''_q(\mathbb{R}_+^N)$, problem (4.39) admits a solution $v=B_0(\lambda)(f,\lambda^{1/2}\rho,\rho)$, and

$$R_{L(Y''_q(\mathbb{R}_+^N),H_q^{2-j}(\mathbb{R}_+^N)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}B_0(\lambda))\mid\lambda\in\Sigma_{\varepsilon,\lambda_0}\})\le r_b(\lambda_0)$$

for $\ell=0,1$ and $j=0,1,2$, where $r_b(\lambda_0)$ is a constant depending on ε, λ_0, N, and q.

Proof: This lemma was proved in Shibata[35] [Lemma 9.3.10], but for the sake of completeness of the paper as much as possible, we give a proof. Let g_1 be a solution of the equation:

$$(\lambda-\Delta)g_1=\mathrm{d}iv f\quad \text{in}\mathbb{R}^N_+,\quad g_1|_{x_N=0}=0,$$

and let g_2 be a solution of the equation:

$$(\lambda-\Delta)g_2=0\quad\text{in } \mathbb{R}^N_+,\quad g_2|_{x_N=0}=\rho.$$

And then, $g=g_1+g_2$ is a solution of Eq. (4.37). To construct g_1 and g_2, we introduce the even extension, f^e, and odd extension, f^o, of a function, f, defined on $\mathbb{R}_+^N$, which are defined by

$$f^e(x)=\begin{cases}f(x',x_N) & x_N>0,\\ f(x',-x_N) & x_N<0,\end{cases}\qquad f^o(x)=\begin{cases}f(x',x_N) & x_N>0,\\ -f(x',-x_N) & x_N<0,\end{cases}\tag{4.40}$$

where $x'=(x_1,\dots,x_{N-1})\in\mathbb{R}^{N-1}$ and $x=(x',x_N)\in\mathbb{R}^N$. Let $f=(f_1,\dots,f_N)^{\mathrm{T}}$. Notice that $(\mathrm{div}f)^o=\sum_{j=1}^{N-1}\partial_j f_j^o+\partial_N f_N^e$. We define g_1 by letting

$$g_1=F_\xi^{-1}\Big[\frac{F[(\mathrm{divf})^o](\xi)}{\lambda+|\xi|^2}\Big]=F_\xi^{-1}\Big[\frac{\sum_{k=1}^{N-1}i\xi_k F[f_k^o](\xi)+i\xi_N F[f_N^e](\xi)}{\lambda+|\xi|^2}\Big].$$

And also, the g_2 is defined by

$$g_2(x)=F_{\xi'}^{-1}[e^{-B_0x_N}\hat\rho(\xi',0)](x')=\frac{\partial h}{\partial x_N},$$

where we have set $B_0=\sqrt{\lambda+|\xi'|^2}$ and $h(x)=-\mathcal{F}_{\xi'}^{-1}[B_0^{-1}e^{-B_0x_N}\hat{\rho}(\xi',0)](x')$. Let v_1 be an N vector of functions defined by

$$v_1=-\mathcal{F}_{\xi}^{-1}[\frac{i\xi\mathcal{F}[g_1](\xi)}{|\xi|^2}]=-\mathcal{F}_{\xi}^{-1}[\frac{\xi(\sum_{k=1}^{N-1}\xi_k\mathcal{F}[f_k^o](\xi)+\xi_N\mathcal{F}[f_N^e](\xi))}{(\lambda+|\xi|^2)|\xi|^2}].$$

We see that $div\ v_1=g_1$ in $\mathbb{R}_+^N$. Moreover, by Lemma 3.2 and Lemma 3.3, there exists an operator family $\mathcal{B}_0^1(\lambda)\in \mathrm{Hol}(\Sigma_\varepsilon,\mathcal{L}(L_q(R_+^N),H_q^2(\mathbb{R}_+^N)^N))$ such that $v_1=\mathcal{B}_0^1(\lambda)f$ and

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}_+^N)^N,H_q^{2-j}(\mathbb{R}_+^N)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{B}_0^1(\lambda))\mid\lambda\in\Sigma_{\varepsilon,\lambda_0}\})\le r_b(\lambda_0)$$

for $\ell=0,1$, $j=0,1,2$, and $\lambda_0>0$, where $r_b(\lambda_0)$ is a constant depending on ε, λ_0, N and q.

Let

$$v_{2j}=\mathcal{F}_{\xi}^{-1}[\frac{\xi_j\xi_N\mathcal{F}[h^e](\xi)}{|\xi|^2}]=-\mathcal{F}_{\xi}^{-1}[\frac{i\xi_j\mathcal{F}[g_2^o](\xi)}{|\xi|^2}],$$

and let $v_2=(v_{21},\dots,v_{2N})^{\mathrm{T}}$, and then we have $div\ v_2=g_2=\partial_N h$ in $\mathbb{R}_+^N$. Since

$$\nabla v_{2j}=\mathcal{F}_{\xi}^{-1}[\frac{\xi\xi_j\mathcal{F}[g_2^o](\xi)}{|\xi|^2}](j=1,\dots,N);$$

$$\nabla^2 v_{2k}=\mathcal{F}_{\xi}^{-1}[\frac{\xi\otimes\xi\mathcal{F}[(\partial_k g_2)^o](\xi)}{|\xi|^2}](k=1,\dots,N-1);$$

$$\partial_k\nabla v_{2N}=\mathcal{F}_{\xi}^{-1}[\frac{\xi\xi_N\mathcal{F}[(\partial_k g_2)^o](\xi)}{|\xi|^2}]\quad(k=1,\dots,N-1);$$

$$\partial_N^2 v_{2N}=\mathcal{F}_{\xi}^{-1}[\frac{\xi_N^2}{|\xi|^2}\mathcal{F}[\partial_N^2 h^e]]=\mathcal{F}_{\xi}^{-1}[\frac{\xi_N^2}{|\xi|^2}\mathcal{F}[\lambda h^e-\Delta' h^e](\xi)],$$

we have

$$\|\lambda v_2\|_{L_q(\mathbb{R}^N)}\le C\|\lambda h\|_{L_q(\mathbb{R}_+^N)},\quad \|\lambda^{1/2}\nabla v_2\|_{L_q(\mathbb{R}^N)}\le C\|\lambda^{1/2}h\|_{L_q(HS)},$$
$$\|\nabla^2 v_2\|_{L_q(\mathbb{R}^N)}\le C(\|\nabla^2 h\|_{L_q(\mathbb{R}_+^N)}+\|\lambda h\|_{L_q(\mathbb{R}_+^N)}).$$

Thus, by Lemma 4.6 and Lemma 4.7, we see that there exists an operator family $\mathcal{B}_0^1(\lambda)$ with

$$\mathcal{B}_0^1(\lambda)\in\mathrm{Hol}(\Sigma_\varepsilon,\mathcal{B}(\mathcal{Z}_q(\mathbb{R}_+^N),H_q^2(\mathbb{R}_+^N)^N))$$

such that $v_2=\mathcal{B}_0^2(\lambda)(\lambda^{1/2}\rho,\rho)$ and

$$\mathcal{R}_{\mathcal{L}(\mathcal{Z}_q(R_+^N),H_q^{2-j}(R_+^N)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{B}_0^2(\lambda))\mid\lambda\in\Sigma_{\varepsilon,\lambda_0}\})\le r_b(\lambda_0)$$

for $\ell=0,1$, $j=0,1,2$, and $\lambda_0>0$, where $r_b(\lambda_0)$ is a constant depending on ε, λ_0, N, and q, and $\mathcal{Z}_q(\mathbb{R}_+^N)$ is the same space as in Lemma 4.8. Since $v=v_1+v_2$ is a solution of Eq. (4.39), setting $\mathcal{B}_0(\lambda)(F_1,G_1,G_2)=\mathcal{B}_0^1(\lambda)F_1+\mathcal{B}_0^2(\lambda)(G_1,G_2)$, we see that $\mathcal{B}_0(\lambda)$ is the required operator, which completes the proof of Lemma 4.12. Let $u_0=\mathcal{B}_0(\lambda)(f,\lambda^{1/2}n_0\cdot h,n_0\cdot h)$, and let $u=u_0+w_0$. We then look for w_0, q, and h satisfying the equations:

$$\begin{cases}\lambda w_0-\mathrm{Div}(\mu\mathrm{D}(w_0)-q\mathrm{I})=\mathrm{f}-\mathrm{f}_0,\quad div w_0=0 & \text{in }\mathbb{R}_+^N,\\ \lambda h+A_\sigma\cdot\nabla' h-U_1\cdot\mathrm{n}_0=d+d_0 & \text{on }\mathbb{R}_0^N,\\ (\mu\mathrm{D}(w_0)-q\mathrm{I})n_0-\delta(\Delta' h)\mathrm{n}_0=\mathrm{h}\text{-}\mathrm{h}_0 & \text{on }\mathbb{R}_0^N,\end{cases}\tag{4.41}$$

where we have set

$$f_0=\lambda u_0-\mathrm{Div}(\mu\mathrm{D}(u_0)),\quad d_0=u_0\cdot n_0,\quad h_0=\mu\mathrm{D}(u_0).$$

We consider the equations:

$$\begin{cases}\lambda\mathrm{U}_1-\mathrm{Div}(\mu\mathrm{D}(\mathrm{U}_1)-P_1\mathrm{I})=\mathrm{F},\quad div\mathrm{U}_1=0 & \text{in }\mathbb{R}_+^N,\\ \partial_N(\mathrm{U}_1\cdot n_0)=0,\quad p_1=0 & \text{on }\mathbb{R}_0^N.\end{cases}\tag{4.42}$$

For $F=(F_1,\dots,F_N)^{\mathrm{T}}\in L_q(\mathbb{R}_+^N)^N$, let $F=(F_1^e,\dots,F_{N-1}^e,F_N^o)^{\mathrm{T}}$. Let $\mathcal{B}_1(\lambda)$ and $\mathcal{P}_1(\lambda)$ be operators acting on $F\in L_q(\mathbb{R}_+^N)^N$ defined by

$$B_1(\lambda)\mathrm{F} = \mathcal{F}_\xi^{-1}[\frac{\mathcal{F}[F](\xi) - \xi\xi\cdot\mathcal{F}[F](\xi)|\xi|^{-2}}{\lambda+\mu|\xi|^2}], \quad P_1(\lambda)\mathrm{F} = \mathcal{F}_\xi^{-1}[\frac{\xi\cdot\mathcal{F}[F](\xi)}{|\xi|^2}].$$

As was seen in Shibata et al.,[40] [40, p.587] or Shibata et al.[41] [41, Proof of Theorem 4.3], $U_1 = B_1(\lambda)F$ and $P_1 = P_1(\lambda)\mathrm{F}$ satisfy Eq. (4.42). Moreover, employing the same argument as in Sect. ??, by Lemma 3.2 and Lemma 3.3, we see that

$$B_1(\lambda) \in \mathrm{Hol}(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(L_q(\mathbb{R}^N_+)^N, H^2_q(R^N_+)^N)), \quad P_1(\lambda) \in \mathrm{Hol}(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(L_q(\mathbb{R}^N_+)^N, \hat{H}^1_q(\mathbb{R}^N_+)))$$

for any $\varepsilon \in (0,\pi/2)$ and $\lambda_0 > 0$, and moreover

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^N, H^{2-j}_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}B_1(\lambda)) \mid \lambda\in\Sigma_{\varepsilon,\lambda_0}\}) \le r_b$$

for $\ell = 0,1$ and $j = 0,1,2$, where r_b is a constant depending on ε, λ_0, m_0, and m_1. In particular, we set

$$u_1 = B_1(\lambda)(f - f), \quad q_1 = P_1(\lambda)(f - f). \tag{4.43}$$

We now let $u = u_0 + u_1 + U_2$ and $q = q_1 + P_2$, and then

$$\begin{cases} \lambda \mathrm{U}_2 - \mathrm{D}iv(\mu \mathrm{D}(\mathrm{U}_2) - P_2\mathrm{I}) = 0, \quad \mathrm{d}iv\mathrm{U}_2 = 0 & \text{in } \mathbb{R}^N_+, \\ \lambda h + A_\sigma\cdot\nabla' h - \mathrm{U}_2\cdot n_0 = d + d_2 & \text{on } \mathbb{R}^N_0, \\ (\mu\mathrm{D}(\mathrm{U}_2) - P_2\mathrm{I})n_0 - \delta(\Delta' h)n_0 = h - h_2 & \text{on } \mathbb{R}^N_0, \end{cases} \tag{4.44}$$

where we have set

$$d_2 = n_0\cdot(u_0+u_1), \quad h_2 = \mu D(u_0+u_1).$$

Thus, for $\mathrm{H} \in H^1_q(\mathbb{R}^N_+)^N$ we consider the equations:

$$\begin{cases} \lambda \mathrm{U}_2 - \mathrm{D}iv(\mu \mathrm{D}(\mathrm{U}_2) - P_2\mathrm{I}) = 0, \quad \mathrm{d}iv\mathrm{U}_2 = 0 & \text{in } \mathbb{R}^N_+, \\ (\mu\mathrm{D}(\mathrm{U}_2) - P_2\mathrm{I})n_0 = \mathrm{H} & \text{on } \mathbb{R}^N_0, \end{cases} \tag{4.45}$$

and then by Theorem 4.3, we see that $U_2 = \mathcal{V}(\lambda)(\lambda^{1/2}\mathbf{H},\mathbf{H})$ is a unique solutions of Eq. (4.45) with some $P_2 \in \hat{H}^1_q(\mathbb{R}^N_+)$. In particular, we set $u_2 = \mathcal{V}(\lambda)(\lambda^{1/2}(h-h_2),(h-h_2))$.

We finally let $u = u_0+u_1+u_2+u_3$ and $q = q_1+q_2+q_3$, and then u_3, q_3 and h are solutions of the equations:

$$\begin{cases} \lambda u_3 - \mathrm{D}iv(\mu \mathrm{D}(u_3) - q_3\mathrm{I}) = 0, \quad \mathrm{d}iv\mathrm{u}_3 = 0 & \text{in } \mathbb{R}^N_+, \\ \lambda h + A_\sigma\cdot\nabla' h - \mathrm{u}_3\cdot \mathrm{n}_0 = d + d_3 & \text{on } \mathbb{R}^N_0, \\ (\mu\mathrm{D}(u_3) - q_3\mathrm{I})n_0 - \delta(\Delta' h)n_0 = 0 & \text{on } \mathbb{R}^N_0, \end{cases} \tag{4.46}$$

where $d_3 = \mathrm{n}_0\cdot(\mathrm{u}_0+\mathrm{u}_1+\mathrm{u}_2)$. By Theorem 4.9, setting $\mathcal{W}(\lambda) = (\mathcal{W}_1(\lambda),\ldots,\mathcal{W}_N(\lambda))^{\mathrm{T}}$, we see that $u_3 = \mathcal{W}(\lambda)(d+d_3)$ and $h = \mathcal{H}_\sigma(\lambda)(d+d_3)$ are unique solutions of Eq. (4.46) with some $q_3 \in \hat{H}^1_q(\mathbb{R}^N_+)$. Since the composition of two $\mathcal{R}$ -bounded operators is also $\mathcal{R}$ bounded as follows from Proposition 3.4, we see easily that given $\varepsilon \in (0,\pi/2)$, there exist $\lambda_1 > 0$ and operator families $\mathcal{A}_0(\lambda)$ and $\mathcal{H}_0(\lambda)$ satisfying (4.4) such that $u = \mathcal{A}_0(\lambda)(f,d,\lambda^{1/2}h,h)$ and $h = \mathcal{H}_0(\lambda)(f,d,\lambda^{1/2}\mathrm{h},\mathrm{h})$ are unique solutions of Eq. (4.3), and moreover the estimate (4.5) holds. This completes the proof of Theorem 4.1.

Problem in a bent half space

Let $\Phi: R^N \to R^N$: $x \to y = \Phi(x)$ be a bijection of C^1 class and let Φ^{-1} be its inverse map. We assume that $\nabla\Phi$ and $\nabla\Phi^{-1}$ have the forms: $\nabla\Phi = \mathcal{A} + B(x)$ and $\nabla\Phi^{-1} = \mathcal{A}_{-1} + B_{-1}(y)$, where $\mathcal{A}$ and $\mathcal{A}_{-1}$ are $N\times N$ orthogonal matrices with constant coefficients and $B(x)$ and $B_{-1}(y)$ are matrices of functions in $C^2(\mathbb{R}^N)$ such that

$$\|(B,B_{-1})\|_{L_\infty(c^N)} \le M_1, \quad \|\nabla(B,B_{-1})\|_{L_\infty(\mathbb{R}^N)} \le C_K, \quad \|\nabla^2(B,B_{-1})\|_{L_\infty(\mathbb{R}^N)} \le M_2. \tag{5.1}$$

Here, C_K is a constant depending on constants K, α, β appearing in Definition 1.1. We choose $M_1 > 0$ small enough and M_2 large enough eventually, and so we may assume that $0 < M_1 \le 1 \le C_K \le M_2$. Let $\Omega_+ = \Phi(\mathbb{R}^N_+)$ and $\Gamma_+ = \Phi(\mathbb{R}^N_0)$. Let n_+ be the unit outer normal to Γ_+. Setting $\Phi^{-1} = (\Phi_{-1,1},\ldots,\Phi_{-1.N})^{\mathrm{T}}$, we see that Γ_+ is represented by $\Phi_{-,N}(y) = 0$, which yields that

$$n_+(x) = -\frac{(\nabla\Phi_{-1,N})\circ\Phi(x)}{(|\nabla\Phi_{-1,N})\circ\Phi(x)|} = -\frac{(a_{N1}+b_{N1}(x),\ldots,a_{NN}+b_{NN}(x))^{\mathrm{T}}}{(\sum_{j=1}^N(a_{Nj}+b_{Nj}(x))^2)^{1/2}} \tag{5.2}$$

where a_{ij} and $b_{ij}(x)$ denote the $(i,j)^{th}$ element of A_{-1} and $(B_{-1}\circ\Phi)(x)$. Obviously, n_+ is defined on $\mathbb{R}^N$ and n_+ denotes the unit outer normal to Γ_+ for $y=\Phi(x',0)\in\Gamma_+$. By (5.1), writing

$$n_+=-(a_{N1},\ldots,a_{NN})^{\mathrm{T}}+b_+(x) \tag{5.3}$$

we see that b_+ is an N-vector defined on $\mathbb{R}^N$, which satisfies the estimates:

$$\|b_+\|_{L_\infty(\mathbb{R}^N)}\le C_N M_1,\quad \|\nabla b_+\|_{L_\infty(\mathbb{R}^N)}\le C_N C_K,\quad \|\nabla^2 b_+\|_{L_\infty(\mathbb{R}^N)}\le C_{M2}. \tag{5.4}$$

We next give the Laplace-Beltrami operator on Γ_+. Let

$$g_{+ij}(x)=\frac{\partial\Phi}{\partial x_i}(x)\cdot\frac{\partial\Phi}{\partial x_j}(x)=\sum_{k=1}^{N}(a_{ik}+b_{ik}(x))(a_{jk}+b_{jk}(x))=\delta_{ij}+\tilde g_{ij}(x)$$

with $\tilde g_{ij}=\sum_{k=1}^{N}(a_{ik}b_{jk}(x)+a_{jk}b_{ik}(x)+b_{ik}(x)b_{jk}(x))$. Since Γ_+ is given by $y_N=\Phi(x',0)$, letting $G(x)$ be an $N\times N$ matrix whose $(i,j)^{th}$ element are $g_{ij}(x)$, we see that $G(x',0)$ is the 1st fundamental matrix of Γ_+. Let $g_+:=\sqrt{\det G}$ and let $g_+^{ij}(x)$ denote the $(i,j)^{th}$ component of the inverse matrix, G^{-1}, of G. By (5.1), we can write

$$g_+=1+\tilde g_+,\quad g_+^{ij}(x)=\delta_{ij}+\tilde g_+^{ij}(x)$$

with

$$\|(\tilde g_+,\tilde g_+^{ij})\|_{L_\infty(\mathbb{R}^N)}\le C_N M_1,\quad \|\nabla(\tilde g_+,\tilde g_+^{ij})\|_{L_\infty(\mathbb{R}^N)}\le C_N C_K,\quad \|\nabla^2(\tilde g_+,\tilde g_+^{ij})\|_{L_\infty(\mathbb{R}^N)}\le C_{M2}. \tag{5.5}$$

The Laplace-Beltrami operator Δ_{Γ_+} is given by

$$(\Delta_{\Gamma_+}f)(y)=\sum_{i,j=1}^{N-1}\frac{1}{g_+(x',0)}\frac{\partial}{\partial x_i}\{g_+(x',0)g_+^{ij}(x',0)\frac{\partial}{\partial x_j}f(\Phi(x',0))\}=\Delta' f(\Phi(x',0))+D_+f \tag{5.6}$$

for $y=\Phi(x',0)\in\Gamma_+$. Where,

$$(D_+f)(y)=\sum_{i,j=1}^{N-1}\tilde g^{ij}(x)\frac{\partial^2 f}{\partial x_i\partial x_j}(x)+\sum_{j=1}^{N-1}g^j(x)\frac{\partial f(x)}{\partial x_j}\quad\text{for } y{=}\Phi(x)$$

with

$$g^j(x)=\frac{1}{g_+(x)}\sum_{i=1}^{N-1}\frac{\partial}{\partial x_i}(g_+(x)g^{ij}(x)).$$

By (5.5)

$$\|D_+f\|_{H_q^1(\mathbb{R}_+^N)}\le C_N M_1\|\nabla^3 f\|_{L_q(\mathbb{R}_+^N)}+C_{M2}\|f\|_{H_q^2(\mathbb{R}_+^N)}. \tag{5.7}$$

We now formulate problem treated in this section. Let y_0 be any point of Γ_+ and let d_0 be a positive number such that

$$\begin{aligned}&|\mu(y)-\mu(x_0)|,|\delta(y)-\delta(y_0)|\le m_1M_1,\quad \text{for any } y\in\overline{\Omega_+}\cap B_{d_0}(y_0);\\ &|A_\sigma(y)-A_\sigma(y_0)|\le m_2M_1\quad \text{foranyy}\in\Gamma_+\cap B_{d_0}(y_0).\end{aligned} \tag{5.8}$$

In addition, μ, δ, and A_σ satisfy the following conditions:

$$\begin{aligned}&m_0\le\mu(y),\delta(y)\le m_1,\quad |\nabla\mu(y)|,|\nabla\delta(y)|\le m_1\quad \text{for any } y\in\overline{\Omega_+},\\ &|A_\sigma(y)|\le m_2\quad \text{for any } y\in\Gamma_+,\quad \|A_\sigma\|_{W_r^{2-1/q}(\Omega_+)}\le m_3\sigma^{-b}\quad \text{for any } \sigma\in(0,1).\end{aligned} \tag{5.9}$$

In view of (1.2), (1.3) and (5.9), to have (5.8) for given $M_1\in(0,1)$ it suffices to choose $d_0>0$ in such a way that $d_0\le M_1$ and $d_0^a\le M_1$. We assume that $N<r<\infty$ and $A_0=0$ according to (1.3). Let $\varphi(y)$ be a function in $C_0^\infty(\mathbb{R}^N)$ which equals 1 for $y\in B_{d_0/2}(y_0)$ and 0 in the outside of $B_{d_0}(y_0)$. We assume that $\|\nabla\varphi\|_{H_\infty^1(\mathbb{R}^N)}\le M_2$. Let

$$\mu_{y_0}(y)=\varphi(y)\mu(y)+(1-\varphi(y))\mu(y_0),\quad \delta_{y_0}(y)=\varphi(y)\delta(y)+(1-\varphi(y))\delta(y_0),$$
$$A_{\sigma,y_0}(y)=\varphi(y)A_\sigma(y)+(1-\varphi(y))A_\sigma(y_0).$$

In the following, C denotes generic constants depending on $m_0,m_1,m_2,m_3,N,\varepsilon$, and q; C_{M_2} denotes generic constants depending on $M_2,m_0,m_1,m_2,m_3,N,\varepsilon$ and q.

Given $v\in H_q^2(\Omega_+)^N$ and $h\in H_q^3(\Omega_+)$, let $K_b(v,h)$ is a unique solution of the weak Dirichlet problem:

$$(\nabla K_b(v,h),\nabla\varphi)_{\Omega_+}=(\mathrm{Div}(\mu_{y_0}D(v))-\nabla\mathrm{div}v,\nabla\varphi)_{\Omega_+}\quad \text{for any}\varphi\in\hat{\mathrm{H}}_{q',0}^1(\Omega_+), \tag{5.10}$$

subject to $K_b(v,h)=<\mu_{y_0}\mathrm{D(v)n_+,n_+}>-\delta_{y_0}\Delta_{\Gamma_+}h-\mathrm{div}\,v$ on Γ_+. We then consider the following equations:

$$\begin{cases} \lambda v - Div(\mu_{y0}\mathrm{D}(v) - K_b(v,h)\mathrm{I}) & = g & \text{in } \Omega_+, \\ \lambda h + A_{\sigma,y_0}\cdot\nabla_{\Gamma_+}h - v\cdot n_+ & = g_d & \text{on } \Gamma_+, \\ (\mu_{y_0}\mathrm{D}(v) - K_b(v,h)\mathrm{I})\mathrm{n}_+ - \delta_{y_0}(\Delta_{\Gamma_+}h)\mathrm{n}_+ & = g_b & \text{on } \Gamma_+. \end{cases} \tag{5.11}$$

The following theorem is a main result in this section.

Theorem 5.1: Let $1<q<\infty$ and $0<\varepsilon<\pi/2$. Let γ_σ be the number defined in Theorem 1.7. Then, there exist $M_1\in(0,1)$, $\tilde\lambda_0\ge 1$ and operator families $A_b(\lambda)$ and $H_b(\lambda)$ with

$$A_b(\lambda)\in \mathrm{Hol}(\Lambda_{\sigma,\tilde\lambda_0\gamma_\sigma}, L(Y_q(\Omega_+),H_q^2(\Omega_+)^N)), H_b(\lambda)\in \mathrm{Hol}(\Lambda_{\sigma,\tilde\lambda_0\gamma_\sigma}, L(Y_q(\Omega_+),H_q^3(\Omega_+)))$$

such that for any $\lambda=\gamma+i\tau\in\Lambda_{\sigma,\tilde\lambda_0\gamma_\sigma}$ and $(g,g_d,g_b)\in Y_q(\Omega_+)$,

$$u=A_b(\lambda)(g,g_d,\lambda^{1/2}g_b,g_b),\quad h=H_b(\lambda)(g,g_d,\lambda^{1/2}g_b,g_b)$$

are unique solutions of Eq. (5.11), and

$$\begin{aligned} &R_{L(Y_q(\Omega_+),H_q^{2-j}(\Omega_+))}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}A_b(\lambda))\mid\lambda\in\Lambda_{\sigma,\tilde\lambda_0\gamma_\sigma}\})\le r_b,\\ &R_{L(Y_q(\Omega_+),H_q^{3-k}(\Omega_+)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^k H_b(\lambda))\mid\lambda\in\Lambda_{\sigma,\tilde\lambda_0\gamma_\sigma}\})\le r_b,\end{aligned} \tag{5.12}$$

for $\ell=0,1$, $j=0,1,2$, and $k=0,1$. Where, r_b is a constant depeding on $m_0,m_1,m_2,m_3,N,\varepsilon$ *and* q but independent of M_1 and M_2, and moreover, $\tilde\lambda_0$ is a constant depending on M_2.

Below, we shall prove Theorem 5.1. By the change of variables $y=\Phi(x)$, we transform Eq. (5.11) to a problem in the half-space. We let

$$y_0=\Phi(x_0),\quad \tilde\mu(x)=\phi(\Phi(x))\mu(\Phi(x)),\quad \tilde\delta(x)=\phi(\Phi(x))\delta(\Phi(x)),\quad \tilde A_\sigma(x)=\phi(\Phi(x))A_\sigma(\Phi(x)).$$

Notice that

$$\mu_{y_0}(\Phi(x))=\mu(y_0)+\tilde\mu(x)-\tilde\mu(x_0),\quad \delta_{y_0}(\Phi(x))=\delta(y_0)+\tilde\delta(x)-\tilde\delta(x_0),$$
$$A_\sigma(\Phi(x',0))=A_\sigma(y_0)+\tilde A_\sigma(x)-\tilde A_\sigma(x_0).$$

We may assume that m_1, m_2, $m_3\le M_2$. Recalling that $\|\nabla\phi\|_{H^1_\infty(\mathbb{R}^N)}\le M_2$, by (5.8) and (5.9) we have

$$\begin{aligned} &|\tilde\mu(x)-\tilde\mu(x_0)|\le m_1M_1,\quad |\tilde\delta(x)-\tilde\delta(x_0)|\le m_1M_1,\quad |\tilde A_\sigma(x)-\tilde A_\sigma(x_0)|\le m_2M_1,\\ &\|(\tilde\mu,\tilde\delta)\|_{L_\infty(R^N)}\le m_1,\quad \|\nabla(\tilde\mu,\tilde\delta)\|_{L_\infty(R^N)}\le C_{M_2},\\ &\|\tilde A_\sigma\|_{L_\infty(R_0^N)}\le m_2,\quad \|\nabla\tilde A_\sigma\|_{W_q^{1-1/q}(R_0^N)}\le C_{M_2}\sigma^{-b}\end{aligned} \tag{5.13}$$

for $\sigma\in(0,1)$.

Since $x=\Phi^{-1}(y)$, we have

$$\frac{\partial}{\partial y_j}=\sum_{k=1}^N(a_{kj}+b_{kj}(x))\frac{\partial}{\partial x_k} \tag{5.14}$$

where $(\nabla\Phi^{-1})(\Phi(x))=(a_{ij}+b_{ij}(x))$. Let

$$g:=\det\nabla\Phi,\quad \tilde g=g-1.$$

By (5.1),

$$\|g\|_{L_\infty(\mathbb{R}^N)}\le C_NM_1,\quad \|\nabla g\|_{L_\infty(\mathbb{R}^N)}\le C_NC_K,\quad \|\nabla^2 g\|_{L_\infty(\mathbb{R}^N)}\le C_{M_2}. \tag{5.15}$$

By the change of variables: $y=\Phi(x)$, the weak Dirichlet problem:

$$(\nabla u,\nabla\varphi)_{\Omega_+}=(k,\nabla\varphi)_{\Omega_+}\quad\text{for any}\,\varphi\in\hat{\mathrm{H}}^1_{q',0}(\Omega_+),$$

subject to $u=k$ on Γ_+, is transformed to the following variational problem:

$$(\nabla v,\nabla\psi)_{\mathbb{R}^N_+}+(B^0\nabla v,\nabla\psi)_{\mathbb{R}^N_+}=(h,\nabla\psi)_{\mathbb{R}^N_+}\quad\text{for any}\,\psi\in\hat{\mathrm{H}}^1_{q',0}(\mathbb{R}^N_+), \tag{5.16}$$

subject to $v=h$, where $h=g(\mathbb{R}A_{-1}+B_{-1}\circ\Phi)k\circ\Phi$ and $h=k\circ\Phi$. Moreover, B^0 is an $N\times N$ matrix whose $(\ell,m)^{th}$ component, $B^0_{\ell m}$, is given by

$$B^0_{\ell m}=g\delta_{\ell m}+g\sum_{j=1}^N(a_{\ell j}b_{mj}(x)+a_{mj}b_{\ell j}(x)+b_{\ell j}b_{mj}(x)).$$

By (5.1), we have

$$\|B^0_{\ell m}\|_{L_\infty(\mathbb{R}^N)} \le C_N M_1, \quad \|\nabla B^0_{\ell m}\|_{L_\infty(\mathbb{R}^N)} \le C_N C_K, \quad \|\nabla^2 B^0_{\ell m}\|_{L_\infty(\mathbb{R}^N)} \le C_{M_2}. \tag{5.17}$$

Lemma 5.2: Let $1 < q < \infty$. Then, there exist an $M_1 \in (0,1)$ and an operator K_1 with

$$K_1 \in L\,(L_q(\mathbb{R}^N_+)^N, H^1_q(\mathbb{R}^N_+) + \hat{H}^1_{q,0}(\mathbb{R}^N_+))$$

such that for any $f \in L_q(\mathbb{R}^N_+)^N$ and $f \in H^1_q(\mathbb{R}^N_+)$, $v = K_1(f, f)$ is a unique solution of the variational problem:

$$(\nabla v, \nabla \psi)_{\mathbb{R}^N_+} + (B^0 \nabla v, \nabla \psi)_{\mathbb{R}^N_+} = (f, \nabla \psi)_{\mathbb{R}^N_+} \quad \text{for any } \psi \in \hat{\mathrm{H}}^1_{q',0}(\mathbb{R}^N_+), \tag{5.18}$$

subject to $v = f$ on $\mathbb{R}^N_0$, which possesses the estimate:

$$\|\nabla v\|_{L_q(\mathbb{R}^N_+)} \le C_{M_2}(\|f\|_{L_q(\mathbb{R}^N_+)} + \|f\|_{H^1_q(\mathbb{R}^N_+)}). \tag{5.19}$$

Proof: We know the unique existence theorem of the variational problem:

$$(\nabla v, \nabla \psi)_{\mathbb{R}^N_+} = (f, \nabla \psi)_{\mathbb{R}^N_+} \quad \text{for any } \psi \in \hat{\mathrm{H}}^1_{q',0}(\mathbb{R}^N_+)$$

subject to $v = f$ on $\mathbb{R}^N_+$. Thus, choosing $M_1 > 0$ small enough in (5.17) and using the Banach fixed point theorem, we can easily prove the lemma. Using the change of the unknown functions: $u = A_{-1} v \circ \Phi$ as well as the change of variable: $y = \Phi(x)$, we will derive the problem in $\mathbb{R}^N_+$ from (5.11). Noting that $A = A_{-1}^{\mathrm{T}}$, by (5.14) we have

$$D_{ij}(v) = \sum_{k,\ell=1}^N a_{ki} a_{\ell j} D_{k\ell}(u) + b^d_{ij} : \nabla u \tag{5.20}$$

with $b^d_{ij} : \nabla u = \sum_{k,\ell=1}^N a_{kj} b_{\ell i} D_{k\ell}(u)$. Setting $b_+(x) = (b_{+1}, \ldots, b_{+N})^{\mathrm{T}}$ in (5.3), by (5.3) we have

$$< D(v) n_+, n_+ > = < D(u) n_0, n_0 > + B^1 : \nabla u \tag{5.21}$$

where we have set

$$B^1 : \nabla u = -2 \sum_{i,j=1}^N a_{ji} b_{+i} D_{jN}(u) + \sum_{i,j,k,\ell=1}^N a_{ki} a_{\ell j} b_{+i} b_{+j} D_{k\ell}(u) + \sum_{i,j=1}^N (b^d_{ij} : \nabla u)(a_{Ni} + b_{+i})(a_{Nj} + b_{+j}).$$

By (5.1), we have

$$\begin{aligned} \|B^1 : \nabla u\|_{L_q(\mathbb{R}^N_+)} &\le C_N M_1 \|\nabla u\|_{L_q(\mathbb{R}^N_+)}, \\ \|B^1 : \nabla u\|_{H^1_q(\mathbb{R}^N_+)} &\le C_N \{M_1 \|\nabla^2 u\|_{L_q(\mathbb{R}^N_+)} + C_K \|u\|_{H^1_q(\mathbb{R}^N_+)}\}. \end{aligned} \tag{5.22}$$

And also,

$$\operatorname{div} v = \operatorname{div} u + B^2 : \nabla u \quad \text{with} \quad B^2 : \nabla u = \sum_{\ell,k=1}^M \left(\sum_{j=1}^N b_{kj} a_{\ell j}\right) \frac{\partial u_\ell}{\partial x_k}. \tag{5.23}$$

By (5.1), we have

$$\begin{aligned} \|B^2 : \nabla u\|_{L_q(\mathbb{R}^N_+)} &\le C_N M_1 \|\nabla u\|_{L_q(\mathbb{R}^N_+)}, \\ \|B^2 : \nabla u\|_{H^1_q(\mathbb{R}^N_+)} &\le C_N \{M_1 \|\nabla^2 u\|_{L_q(\mathbb{R}^N_+)} + C_K \|u\|_{H^1_q(\mathbb{R}^N_+)}\}. \end{aligned} \tag{5.24}$$

By (5.20), we have

$$A_{-1} \mathrm{Div}(\mu_{y_0} D(\mathrm{v})) = \mathrm{Div}(\mu(y_0) \mathrm{D}(\mathrm{u})) + R^1 : u \tag{5.25}$$

with $R^1 : u = (R^1 : u|_1, \ldots, R^1 : u|_N)^{\mathrm{T}}$, and

$$R^1 : u|_s = \sum_{k=1}^N \frac{\partial}{\partial x_k} \{(\tilde{\mu}(x) - \tilde{\mu}(x_0)) D_{sk}(u)\} + \sum_{i,j,k=1}^N a_{si} a_{kj} \frac{\partial}{\partial x_k} (\tilde{\mu}(x) b^d_{ij} : \nabla u) + \sum_{j,k,\ell,m=1}^N a_{\ell j} b_{kj} \frac{\partial}{\partial x_k} (\tilde{\mu} D_{s\ell}(u)) + \sum_{i,j,k=1}^N a_{si} b_{kj} \frac{\partial}{\partial x_k} (\tilde{\mu} b^d_{ij} : \nabla u)$$

By (5.1) and (5.13),

$$\|R^1 : u\|_{L_q(\mathbb{R}^N_+)} \le C_N m_1 M_1 \|\nabla^2 u\|_{L_q(\mathbb{R}^N_+)} + C_{M_2} \|u\|_{H^1_q(\mathbb{R}^N_+)}. \tag{5.26}$$

And also, by (5.20)

$$(A_{-1}+B_{-1}\circ\Phi^{-1})\mathrm{D}iv(\mu_{y_0}\mathrm{D}(v))=\mathrm{D}iv(\mu(y_0)\mathrm{D}(u))+\mathcal{R}^2:u$$

with $\mathcal{R}^2:u=(\mathcal{R}^2:u|_1,\ldots,\mathcal{R}^2:u|_N)$ and

$$\mathcal{R}^2:u|_s=\mathcal{R}^1:u|_s+\sum_{i,j,k=1}^N b_{si}(a_{kj}+b_{kj})\frac{\partial}{\partial x_k}[\tilde\mu(x)\{\sum_{\ell,m=1}^N a_{\ell i}a_{mj}D_{\ell m}(u)+b_{ij}^d:\nabla u\}].$$

By (5.1)

$$\|\mathcal{R}^2:u\|_{L_q(\mathbb{R}_+^N)}\le C_N m_1(M_1\|\nabla^2 u\|_{L_q(\mathbb{R}_+^N)}+C_{M_2}\|u\|_{H_q^1(\mathbb{R}_+^N)}. \tag{5.27}$$

And also, we have

$$(A_{-1}+B_{-1}\circ\Phi^{-1})(\nabla\mathrm{d}iv\, v)\circ\Phi=\nabla\mathrm{d}iv\, u+\mathcal{R}^3:u$$

with $\mathcal{R}^3:u=(\mathcal{R}^3:u|_1,\ldots,\mathcal{R}^3:u|_N)$ and

$$\mathcal{R}^3:u|_s=\frac{\partial}{\partial x_s}(\mathcal{B}^2:\nabla u)+\sum_{k=1}^N\{\sum_{i=1}^N(a_{si}b_{ki}+b_{si}(a_{ki}+b_{ki}))\}\frac{\partial}{\partial x_k}(\mathrm{d}iv u+\mathcal{B}^2:\nabla u).$$

By (5.1)

$$\|\mathcal{R}^3:u\|_{L_q(\mathbb{R}_+^N)}\le C_N m_1(M_1\|\nabla^2 u\|_{L_q(\mathbb{R}_+^N)}+C_K\|u\|_{H_q^1(\mathbb{R}_+^N)}). \tag{5.28}$$

Let

$$f(u):=\mathcal{g}(A_{-1}+B_{-1}\circ\Phi)(\mathrm{D}iv(\mu_{y_0}D(v))-\nabla\mathrm{d}iv v)\circ\Phi,$$

and then

$$f(u)=\mathrm{D}iv(\mu(y_0)D(u))-\nabla\mathrm{d}iv u+\mathcal{R}^4:u$$

with $\mathcal{R}^4:u=(\mathcal{R}^4:u|_1,\ldots\mathcal{R}^4:u|_N)$ and

$$\mathcal{R}^4:u|_s=\mathcal{g}(\mathrm{D}iv(\mu(y_0)D(u))-\nabla\mathrm{d}iv u)+\mathcal{g}\mathcal{R}^2:u-\mathcal{g}\mathcal{R}^3:u.$$

By (5.1), (5.13), (5.15), (5.26), (5.27), and (5.28),

$$\|\mathcal{R}^4:u\|_{L_q(\mathbb{R}_+^N)}\le C_N(m_1+1)M_1\|\nabla^2 u\|_{L_q(\mathbb{R}_+^N)}+C_{M_2}\|u\|_{H_q^1(\mathbb{R}_+^N)}. \tag{5.29}$$

In view of (5.6), (5.21) and (5.23), setting

$$\rho=h\circ\Phi,$$

$$f(u,\rho)=<(\tilde\mu(x)-\tilde\mu(x_0))D(u)n_0,n_0>+\tilde\mu\mathcal{B}^1:\nabla u-(\tilde\delta(x)-\tilde\delta(x_0))\Delta'\rho-\tilde\delta(x)\mathcal{D}_+\rho-\mathcal{B}^2:\nabla u,$$

we have

$$<\mu_{y_0}\mathrm{D}(v)n_+,n_+>-\delta_{y_0}\Delta_{\Gamma_+}h-\mathrm{d}iv v=<\mu(y_0)\mathrm{D}(\mathrm{u})n_0,n_0>-\delta(y_0)\Delta'\rho-\mathrm{d}iv u+f(u,\rho).$$

Thus, $K_1(u,\rho)=K_b(v,h)\circ\Phi$ satisfies the variational equation:

$$(\nabla K_1(u,\rho),\nabla\psi)_{\mathbb{R}_+^N}+(\mathcal{B}^0\nabla K_1(u,\rho),\nabla\psi)_{\mathbb{R}_+^N}=(\mathrm{D}iv(\mu(y_0)\mathrm{D}(\mathrm{u}))-\nabla\mathrm{d}iv\mathrm{u}+\mathcal{R}^4:u,\nabla\psi)_{\mathbb{R}_+^N}$$

for any $\psi\in\hat H^1_{q',0}(\mathbb{R}_+^N)$, subject to $K_1(u,\rho)=<\mu(y_0)D(u)n_0,n_0>-\delta(y_0)\Delta'\rho-\mathrm{d}iv u+f(u,\rho)$ on $\mathbb{R}_0^N$.

Let $\tilde K_0(u,\rho)\in H_q^1(\mathbb{R}_+^N)+\hat H^1_{q,0}(\mathbb{R}_+^N)$ be a unique solution of the weak Dirichlet problem:

$$(\nabla\tilde K_0(u,\rho),\nabla\psi)_{\mathbb{R}_+^N}=(\mathrm{D}iv(\mu(y_0)\mathrm{D}(\mathrm{u}))-\nabla\mathrm{d}iv u,\nabla\psi)_{\mathbb{R}_+^N}\quad\text{for any}\,\psi\in\hat{\mathrm{H}}^1_{\mathrm{q}',0}(\mathbb{R}_+^N),$$

subject to $\tilde K_0(u,\rho)=<\mu(y_0)\mathrm{D}(\mathrm{u})n_0,n_0>-\delta(y_0)\Delta'\rho-\mathrm{d}iv u$ on $\mathbb{R}_0^N$. Setting $K_1(u,\rho)=\tilde K_0(u,\rho)+K_2(u,\rho)$, we then see that $K_2(u,\rho)$ satisfies the variational equation:

$$(\nabla K_2(u,\rho),\nabla\psi)_{\mathbb{R}_+^N}+(\mathcal{B}^0\nabla K_2(u,\rho),\nabla\psi)_{\mathbb{R}_+^N}=(\mathcal{R}^4:u-\mathcal{B}^0\nabla\tilde K_0(u,\rho),\nabla\psi)_{\mathbb{R}_+^N}$$

for any $\varphi\in\hat H^1_{q',0}(\mathbb{R}_+^N)$, subject to $K_2(u,\rho)=f(u,\rho)$ on $\mathbb{R}_0^N$. In view of Lemma 5.2, we have

$$K_2(u,\rho)=\mathcal{K}_1(\mathcal{R}^4:u-\mathcal{R}^0\nabla\tilde K_0(u,\rho),f(u,\rho)).$$

By Lemma 5.2, (5.17), (5.22), (5.24), (5.7), and (5.29), we have

$$\begin{aligned}&\|\nabla K_2(u,\rho)\|_{L_q(\mathbb{R}_+^N)}\\&\le C_N(1+m_1)M_1(\|\nabla^2 u\|_{L_q(\mathbb{R}_+^N)}+\|\nabla^3\rho\|_{L_q(\mathbb{R}_+^N)})+C_{M_2}(\|u\|_{H_q^1(\mathbb{R}_+^N)}+\|\rho\|_{H_q^2(\mathbb{R}_+^N)}).\end{aligned} \tag{5.30}$$

Since

$$A_{-1}\nabla K_b(v,h)|_s = \sum_{i,k=1}^N a_{si}(a_{ki}+b_{ki})\frac{\partial}{\partial x_k}K_1(\mathrm{u},\rho)$$

$$=\frac{\partial}{\partial x_s}\tilde K_0(u,\rho)+\sum_{k=1}^N(\sum_{i=1}^N a_{si}b_{ki})\frac{\partial}{\partial x_k}\tilde K_0(u,\rho)+\sum_{k=1}^N(\delta_{ks}+\sum_{i=1}^N a_{si}b_{ki})\frac{\partial}{\partial x_k}K_2(u,\rho),$$

by (5.25) we see that the first equation of Eq.(5.11) is transformed to

$$\lambda u-\mathrm{D}iv(\mu(y_0)D(u)-\tilde K_0(u,\rho)I)+R^5(u,\rho)=h \quad \text{in } \mathbb{R}^N_+,$$

where $h=A_{-1}g\circ\Phi$, $R^5(u,\rho)=(R^5(u,\rho)|_1,\ldots,R^5(u,\rho)|_N)$, and

$$R^5(u,\rho)|_s=-R^1:u|_s+\sum_{k=1}^N(\sum_{i=1}^N a_{si}b_{ki})\frac{\partial}{\partial x_k}\tilde K_0(u,\rho)+\sum_{k=1}^N(\delta_{ks}+\sum_{i=1}^N a_{si}b_{ki})\frac{\partial}{\partial x_k}K_2(u,\rho).$$

By (5.3), we have

$$v\cdot n_+=-A_{-1}^{\mathrm{T}}u\cdot(a_{N1},\ldots,a_{NN})^{\mathrm{T}}+A_{-1}^{\mathrm{T}}u\cdot b_+=u\cdot n_0+u\cdot(A_{-1}b_+),$$

and so the second equation of Eq.(5.11) is transformed to

$$\lambda\rho+A_\sigma(y_0)\cdot\nabla'\rho-u\cdot n_0+R_\sigma^6(u,\rho)=h_d$$

with $h_d=g_d\circ\Phi$ and

$$R_0^6(u,\rho)=-u\cdot(A_{-1}b_+) \quad \text{for } \sigma=0,$$

$$R_\sigma^6(u,\rho)=(\tilde A_\sigma(x)-\tilde A_\sigma(x_0))\nabla'\rho-u\cdot(A_{-1}b_+) \quad \text{for} \sigma\in(0,1).$$

By (5.3) and (5.20), we have $A_{-1}\mu_{y_0}D(v)n_+=\mu(y_0)D(u)n_0+R_1^7(u)$, where $R_1^7(u)$ is an N- vector of functions whose s^{th} component, $R_1^7(u)|_s$, is defined by

$$R_1^7(u)|_s=-(\tilde\mu(x)-\tilde\mu(x_0))D_{sN}(u)$$

$$+(\mu(y_0)+\tilde\mu(x)-\tilde\mu(x_0))\sum_{i,j=1}^N(a_{ij}b_{+j}D_{si}(u)+a_{si}b_{ij}^d:\nabla u(-a_{Nj}+b_{+j})).$$

By (5.3),

$$A_{-1}K_b(v,h)n_+=\tilde K_0(u,\rho)n_0+\tilde K_0(u,\rho)A_{-1}b_++K_2(u,\rho)(n_0+A_{-1}b_+).$$

By (5.6),

$$A_{-1}\delta_{y_0}(\Delta_{\Gamma_+}h)n_+=\delta(y_0)(\Delta'\rho)n_0+(\tilde\delta(x)-\tilde\delta(x_0))(\Delta'\rho)n_0$$

$$+\tilde\delta(x)\{(\Delta'\rho)(A_{-1}b_+)+(D_+\rho)(n_0+A_{-1}b_+)\}.$$

Putting formulas above together yields that the third equation of Eq.(5.11) is transformed to the equation:

$$(\mu(y_0)D(u)-\tilde K_0(u,\rho)I)n_0-\delta(y_0)(\Delta'\rho)n_0+R^7(u,\rho)=h_b \quad \text{on } \mathbb{R}^N_0,$$

where $h_b=A_{-1}g_b\circ\Phi$, and

$$R^7(u,\rho)=R_1^7(u,\rho)-\tilde K_0(u,\rho)(A_{-1}b_+)-K_2(u,\rho)(n_0+A_{-1}b_+)$$

$$-(\tilde\delta(x)-\tilde\delta(x_0))(\Delta'\rho)n_0-\tilde\delta(x)\{(\Delta'\rho)(A_{-1}b_+)+(D_+\rho)(n_0+A_{-1}b_+)\}.$$

Summing up, we have seen that Eq.(5.11) is transformed to the following equations:

$$\begin{cases}\lambda u-\mathrm{Div}(\mu(y_0)\mathrm{D(u)}-\tilde K_0(u,\rho)\mathrm{I})+R^5(\mathrm{u},\rho) & =h \quad \text{in } \mathbb{R}^N_+,\\ \lambda\rho+A_\sigma(y_0)\cdot\nabla'\rho-u\cdot n_0+R_\sigma^6(\mathrm{u},\rho) & =h_d \quad \text{on } \mathbb{R}^N_0,\\ (\mu(y_0)\mathrm{D(u)}-\tilde K_0(u,\rho)\mathrm{I})n_0-\delta(y_0)(\Delta'\rho)n_0+R^7(\mathrm{u},\rho) & =h_b \quad \text{on } \mathbb{R}^N_0,\end{cases} \tag{5.31}$$

where $h=A_{-1}g\circ\Phi$, $h_d=g_d\circ\Phi$, $h_d=A_{-1}g_d\circ\Phi$, and $R^5(u,\rho)$, $R_\sigma^6(u,\rho)$ and $R^7(u,\rho)$ are linear in u and ρ and satisfy the estimates:

$$
\begin{aligned}
\|R^5(u,\rho)\|_{L_q(\mathbb{R}^N_+)} &\le CM_1(\|\nabla^2 u\|_{L_q(\mathbb{R}^N_+)} + \|\nabla^3\rho\|_{L_q(\mathbb{R}^N_+)}) + C_{M_2}(\|u\|_{H^1_q(\mathbb{R}^N_+)} + \|\rho\|_{H^2_q(\mathbb{R}^N_+)})\},\\
\|R^6_0(u,\rho)\|_{W^{2-1/q}_q(\mathbb{R}^N_0)} &\le CM_1\|\nabla^2 u\|_{L_q(\mathbb{R}^N_+)} + C_{M_2}\|u\|_{H^1_q(\mathbb{R}^N_+)},\\
\|R^6_\sigma(u,\rho)\|_{W^{2-1/q}_q(\mathbb{R}^N_0)} &\le CM_1(\|\nabla^2 u\|_{L_q(\mathbb{R}^N_+)} + \|\nabla^3\rho\|_{L_q(\mathbb{R}^N_+)}) + C_{M_2}(\|u\|_{H^1_q(\mathbb{R}^N_+)} + \sigma^{-b}\|\rho\|_{H^2_q(\mathbb{R}^N_+)}\}, \qquad (5.32)\\
\|R^7(u,\rho)\|_{L_q(\mathbb{R}^N_+)} &\le CM_1(\|\nabla u\|_{L_q(\mathbb{R}^N_+)} + \|\nabla^2\rho\|_{L_q(\mathbb{R}^N_+)}) + C_{M_2}(\|u\|_{L_q(\mathbb{R}^N_+)} + \|\rho\|_{H^1_q(\mathbb{R}^N_+)}),\\
\|R^7(u,\rho)\|_{H^1_q(\mathbb{R}^N_+)} &\le CM_1(\|\nabla^2 u\|_{L_q(\mathbb{R}^N_+)} + \|\nabla^3\rho\|_{L_q(\mathbb{R}^N_+)}) + C_{M_2}(\|u\|_{H^1_q(\mathbb{R}^N_+)} + \|\rho\|_{H^2_q(\mathbb{R}^N_+)}).
\end{aligned}
$$

Here and in the following, C denotes a generic constant depending on N, q, m_1, and m_2 and C_{M_2} a generic constant depending on N, q, m_1, m_2, m_3 and M_2. By Theorem 4.1, there exists a large numbger λ_0 and operator families $A_0(\lambda)$ and $H_0(\lambda)$ with

$$A_0(\lambda) \in \mathrm{Hol}(\Lambda_{\sigma,\lambda_0}, L(Y(\mathbb{R}^N_+), H^2_q(\mathbb{R}^N_+)^N)), \quad H_0(\lambda) \in \mathrm{Hol}(\Lambda_{\sigma,\lambda_0}, L(Y(\mathbb{R}^N_+), H^3_q(\mathbb{R}^N_+)))$$

such that for any $\lambda \in \Lambda_{\sigma,\lambda_0}$ and $(f,d,h) \in Y_q(\mathbb{R}^N_+)$, u and ρ with

$$u = A_0(\lambda)F_\lambda(f,d,h), \quad \rho = H_0(\lambda)F_\lambda(f,d,h),$$

where $F_\lambda(f,d,h) = (f,d,\lambda^{1/2}h,h)$, are unique solutions of the equations:

$$
\begin{cases}
\lambda u - \mathrm{Div}(\mu(y_0)\mathrm{D(u)} - \tilde{K}_0(u,\rho)\mathrm{I}) & = \mathrm{f} \quad \text{in } \mathbb{R}^N_+,\\
\lambda\rho + A_\sigma(y_0)\cdot\nabla'\rho - u\cdot n_0 & = d \quad \text{on } \mathbb{R}^N_0,\\
(\mu(y_0)\mathrm{D(u)} - \tilde{K}_0(u,\rho)\mathrm{I})n_0 - \delta(y_0)(\Delta'\rho)n_0 & = \mathrm{h}, \quad \text{on } \mathbb{R}^N_0,
\end{cases}
$$

and

$$R_{L(Y(\mathbb{R}^N_+), H^{2-j}_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^s(\lambda^{j/2}A_0(\lambda)) \mid \lambda \in \Lambda_{\sigma,\lambda_0}\}) \le r_b,$$

$$R_{L(Y(\mathbb{R}^N_+), H^{3-k}_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial_\tau)^s(\lambda^k H_0(\lambda)) \mid \lambda \in \Lambda_{\sigma,\lambda_0}\}) \le r_b$$

for $s = 0,1$, $j = 0,1,2$, and $k = 0,1$. Where, r_b is a constant depending on ε, N, m_1, and m_2.

Let $u = A_0(\lambda)F_\lambda(h,h_d,h_b)$ and $\rho = H_0(\lambda)F_\lambda(h,h_d,h_b)$ in (5.31). Then, Eq.(5.31) is rewritten as

$$
\begin{cases}
\lambda u - \mathrm{Div}(\mu(y_0)D(u) - \tilde{K}_0(u,\rho)I) + R^5(u,\rho) & = h + R^8(\lambda)F_\lambda(h,h_d,h_b) \quad \text{in } \mathbb{R}^N_+,\\
\lambda\rho + A_\sigma\cdot\nabla'\rho - u\cdot n_0 + R^6_\sigma(u,\rho) & = h_d + R^8_d(\lambda)F_\lambda(h,h_d,h_b) \quad \text{in } \mathbb{R}^N_+,\\
(\mu(y_0)\mathrm{D(u)} - \tilde{K}_0(u,\rho)\mathrm{I})n_0 - \delta(y_0)(\Delta'\rho)n_0 + R^7(u,\rho) & = h_b + R^8_b(\lambda)F_\lambda(h,h_d,h_b) \quad \text{on } \mathbb{R}^N_0,
\end{cases} \qquad (5.33)
$$

where we have set

$$
\begin{aligned}
R^8(\lambda)(F_1,F_2,F_3,F_4) &= R^5(A_0(\lambda)(F_1,F_2,F_3,F_4), H_0(\lambda)(F_1,F_2,F_3,F_4)),\\
R^8_d(\lambda)(F_1,F_2,F_3,F_4) &= R^6_\sigma(A_0(\lambda)(F_1,F_2,F_3,F_4), H_0(\lambda)(F_1,F_2,F_3,F_4)),\\
R^8_b(\lambda)(F_1,F_2,F_3,F_4) &= R^7(A_0(\lambda)(F_1,F_2,F_3,F_4), H_0(\lambda)(F_1,F_2,F_3,F_4)).
\end{aligned}
$$

Let

$$R^9(\lambda)F = (R^8(\lambda)F, R^8_d(\lambda)F, R^8_b(\lambda)F)$$

for $F = (F_1,F_2,F_3,F_4) \in Y_q(\mathbb{R}^N_+)$. Notice that

$$R^9(\lambda)F = (R^8(\lambda)F, R^8_d(\lambda)F, \lambda^{1/2}R^8_b(\lambda)F, R^8_b(\lambda)F) \in Y_q(\mathbb{R}^N_+) \quad \mathrm{for F=(F_1,F_2,F_3,F_4)} \in Y_q(\mathbb{R}^N_+),$$

and that the right side of Eq.(5.33) is written as $(h,h_d,h_b) + F^9(\lambda)F_\lambda(h,h_d,h_b)$. By (5.32), (5.4), Proposition 3.4, and Theorem 4.1, we have

$$R_{L(Y_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^\ell(F_\lambda R^9(\lambda)) \mid \lambda \in \Sigma_{\sigma,\lambda_1}\}) \le CM_1 + C_{M_2}(\lambda_1^{-1/2} + \lambda_1\gamma_\sigma) \qquad (5.34)$$

for any $\lambda_1 \ge \lambda_0$. Here and in the following, C denotes a generic constant depending on N, ε, m_1, m_2, and C_K, and C_{M_2} denotes a generic constant depending on N, ε, m_1, m_2, m_3, C_K, and M_2. Choosing M_1 so small that $C_N M_1 \le 1/4$ and choosing $\lambda_1 > 0$ so large that $C_{M_2}\lambda_1^{-1/2} \le 1/8$ and $C_{M_2}\lambda_1^{-1}\gamma_\sigma \le 8$, by (5.34) we have

$$R_{L(Y_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^\ell(F_\lambda R^9(\lambda)) \mid \lambda \in \Lambda_{\sigma,\lambda_1}\}) \le 1/2 \qquad (5.35)$$

for $\ell = 0,1$. Since $\gamma_\sigma \geq 1$ and we may assume that $C_{M_2} \geq 1$, if $\lambda_1 \geq 64 C_{M_2}^2 \gamma_\sigma$, then $C_{M_2}\lambda_1^{-1/2} \leq 1/8$ and $C_{M_2}\lambda_1^{-1}\gamma_\sigma \leq 1/8$. Recall that for $F = (F_1, F_2, F_3, F_4) \in Y_q(\mathbb{R}_+^N)$ and $(h, h_d, h_b) \in Y_q(\mathbb{R}_+^N)$,

$$
\begin{aligned}
\|(F_1,F_2,F_3,F_4)\|_{Y_q(\mathbb{R}_+^N)} &= \|(F_1,F_3)\|_{L_q(\mathbb{R}_+^N)} + \|F_2\|_{W_q^{2-1/q}(\mathbb{R}_0^N)} + \|F_4\|_{H_q^1(\mathbb{R}_+^N)},\\
\|(h,h_d,h_b)\|_{X_q(\mathbb{R}_+^N)} &= \|h\|_{L_q(\mathbb{R}_+^N)} + \|h_d\|_{W_q^{2-1/q}(\mathbb{R}_0^N)} + \|h_b\|_{H_q^1(\mathbb{R}_+^N)}
\end{aligned}
\tag{5.36}
$$

(cf. Remark 2.2, where Ω should be replaced by $\mathbb{R}_+^N$). By (5.35) we have

$$
\left\| F_\lambda(\mathcal{R}^9(\lambda)F_\lambda(h,h_d,h_b))Y\right\|_{q(\mathbb{R}_+^N)} \leq (1/2)\|F_\lambda(h,h_d,h_b)\|_{Y_q(\mathbb{R}_+^N)}. \tag{5.37}
$$

In view of (5.36), when $\lambda \neq 0$, $\|P_\lambda(h,h_d,h_b)\|_{Y_q(\mathbb{R}_+^N)}$ is an equivalent norm to $\|(h,h_d,h_b)\|_{X_q(\mathbb{R}_+^N)}$. Thus, by (5.37) $(I + \mathcal{R}^9(\lambda)F_\lambda)^{-1} = \sum_{j=1}^\infty (-\mathcal{R}^9(\lambda)F_\lambda)^j$ exists in $\mathcal{L}(X_q(\mathbb{R}_+^N))$. Setting

$$
u = \mathcal{A}_0(\lambda)F_\lambda(I+\mathcal{R}^9(\lambda)F_\lambda)^{-1}(h,h_d,h_b), \quad \rho = \mathcal{H}_0(\lambda)F_\lambda(I+\mathcal{R}^9(\lambda)F_\lambda)^{-1}(h,h_d,h_b) \tag{5.38}
$$

by (5.33) we see that u and ρ are solutions of Eq.(5.31). In view of (5.33), $(I + F_\lambda \mathcal{R}^9(\lambda))^{-1} = \sum_{j=0}^\infty (-F_\lambda \mathcal{R}^9(\lambda))^j$ exists in $\mathcal{L}(Y_q(\mathbb{R}_+^N))$, and

$$
\mathcal{R}_{\mathcal{L}(Y_q(\mathbb{R}_+^N))}(\{(\tau\partial_\tau)^\ell (I + F_\lambda \mathcal{R}^9(\lambda))^{-1} \mid \lambda \in \Lambda_{\sigma,\lambda_1}\}) \leq 4 \tag{5.39}
$$

for $\ell = 0,1$. Since

$$
F_\lambda(\mathrm{I}+\mathcal{R}^9(\lambda)F_\lambda)^{-1} = F_\lambda\sum_{j=0}^\infty(-\mathcal{R}^9(\lambda)F_\lambda)^j = (\sum_{j=0}^\infty(-F_\lambda\mathcal{R}^9(\lambda))^j)F_\lambda = (\mathrm{I}+F_\lambda\mathcal{R}^9(\lambda))^{-1}F_\lambda,
$$

defining operators $\mathcal{A}_1(\lambda)$ and $\mathcal{H}_1(\lambda)$ acting on $F = (F_1, F_2, F_3, F_4) \in Y_q(\mathbb{R}_+^N)$ by

$$
\mathcal{A}_1(\lambda)F = \mathcal{A}_0(\lambda)(I + F_\lambda\mathcal{R}^9(\lambda))^{-1}F, \quad \mathcal{H}_1(\lambda)F_1 = \mathcal{H}_0(\lambda)(\mathrm{I}+F_\lambda\mathcal{R}^9(\lambda))^{-1}F,
$$

by (5.38) $u = \mathcal{A}_1(\lambda)F_\lambda(h,h_d,h_b)$ and $\rho = \mathcal{H}_1(\lambda)F_\lambda(h,h_d,h_b)$ are solutions of Eq.(5.31). Moreover, by (5.39) and Theorem 4.1

$$
\begin{aligned}
\mathcal{R}_{\mathcal{L}(Y_q(\mathbb{R}_+^N), H_q^{2-j}(\mathbb{R}_+^N)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{R}_1(\lambda)) \mid \lambda \in \Lambda_{\sigma,\lambda_1\gamma_\sigma}\}) &\leq 4r_b,\\
\mathcal{R}_{\mathcal{L}(Y_q(\mathbb{R}_+^N), H_q^{3-k}(\mathbb{R}_+^N)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{k}\mathcal{H}_1(\lambda)) \mid \lambda \in \Lambda_{\sigma,\lambda_1\gamma_\sigma}\}) &\leq 4r_b,
\end{aligned}
\tag{5.40}
$$

for $\ell = 0,1$, $j = 0,1,2$ and $k = 0,1$. Recalling that

$$
v = \mathcal{A}_{-1}^{\mathrm{T}} u \circ \Phi^{-1}, h = \rho \circ \Phi^{-1}, h = \mathcal{A}_{-1} g \circ \Phi, h_d = g_d \circ \Phi, h_d = \mathcal{A}_{-1} g_d \circ \Phi,
$$

we define operators $\mathcal{A}_b(\lambda)$ and $\mathcal{H}_b(\lambda)$ acting on $F = (F_1, F_2, F_3, F_4) \in Y_q(\Omega_+)$ by

$$
\begin{aligned}
\mathcal{A}_b(F_1,F_2,F_3,F_4) &= \mathcal{A}_{-1}^{\mathrm{T}}[\mathcal{A}_1(\lambda)(\mathcal{A}_{-1}F_1\circ\Phi, F_2\circ\Phi, \mathcal{A}_{-1}F_3\circ\Phi, F_4\circ\Phi)]\circ\Phi^{-1},\\
\mathcal{H}_b(F_1,F_2,F_3,F_4) &= [\mathcal{H}_1(\lambda)(\mathcal{H}_{-1}F_1\circ\Phi, F_2\circ\Phi, \mathcal{H}_{-1}F_3\circ\Phi, F_4\circ\Phi)]\circ\Phi^{-1}.
\end{aligned}
$$

Obviously, given any $(g, g_d, g_b) \in Y_q(\Omega_+)$, $u = \mathcal{A}_b(\lambda)F_\lambda(g,g_d,g_b)$ and $h = \mathcal{H}_b(\lambda)F_\lambda(g,g_d,g_b)$ are solutions of Eq.(5.11). From (5.1) we have

$$
\begin{aligned}
&\|g\circ\Phi^{-1}\|_{H_q^\ell(\Omega_+)} \leq C_K\|g\|_{H_q^\ell(\mathbb{R}_+^N)} \quad \text{for } \ell=0,1,2,\\
&\|\nabla^3(g\circ\Phi^{-1})\|_{L_q(\Omega_+)} \leq C_K\|\nabla^2 g\|_{H_q^1(\mathbb{R}_+^N)} + C_{M_2}\|\nabla g\|_{L_q(\mathbb{R}_+^N)},\\
&\|h\circ\Phi\|_{H_q^\ell(\mathbb{R}_+^N)} \leq C_K\|h\|_{H_q^\ell(\Omega_+)} \quad \text{for } h\ \ell=0,1,2.
\end{aligned}
$$

and so, in view of (5.12) we can choose $\tilde{\lambda}_0 \geq \lambda_1$ suitably large such that $\mathcal{A}_b(\lambda)$ and $\mathcal{H}_b(\lambda)$ satisfy (5.12). This completes the existence part of Theorem 5.1.

The uniqueness can be proved by showing a priori estimates of solutions of Eq. (5.11) in the same manner as in the proof of Theorem 3.5. This completes the proof of Theorem 5.1.

Proof of theorem 2.1

Some preparation for the proof of theorem 2.1

First, we state several properties of uniform C^k domains ($k = 2,3$).

Proposition 6.1 Let $k = 2$ or 3 and let Ω be a uniformly C^k domain in $\mathbb{R}^N$. Let M_1 be any number in $(0,1)$. Then, there exist two positive constants M_2 and r_0 depending on M_1, at most countably many N-vector of functions $\Phi_j \in C^k(\mathbb{R}^N)^N$ and points $x_j^0 \in \Omega$ and $x_j^1 \in \Gamma$ such that the following assertions hold:

1) The maps: $\mathbb{R}^N \ni x \mapsto \Phi_j(x) \in \mathbb{R}^N$ are bijections satisfying the following conditions: $\nabla\Phi_j^i = A_j^i + B_j^i$, $\nabla(\Phi_j^i)^{-1} = A_{j,-}^i + B_{j,-}^i$, where A_j^i and $A_{j,-}^i$ are $N \times N$ constant orthonormal matrices, and B_j^i and $B_{j,-}^i$ are $N \times N$ matrices of $C^{k-1}(\mathbb{R}^N)$ functions defined on $\mathbb{R}^N$ which satisfy the conditions: $\|(B_j^i, B_{j,-}^i)\|_{L_\infty(\mathbb{R}^N)} \le M_1$ and $\|\nabla(B_j^i, B_{j,-}^i)\|_{L_r(\mathbb{R}^N)} \le C_K$, where C_K is a constant depending on constans α, β and K appearing in Definition 1.1 but independent of M_1. Moreover, if $k = 3$, then $\|\nabla^2(B_j^i, B_{j,-}^i)\|_{L_\infty(\mathbb{R}^N)} \le M_2$.

2) $$\Omega = (\bigcup_{i=1}^\infty B_{r_0}(x_j^0)) \cup (\bigcup_{j=1}^\infty (\Phi_j(\mathbb{R}_+^N) \cap B_{r_0}(x_j^1))), \quad B_{r_0}(x_j^0) \subset \Omega$$
$$\Phi_j(\mathbb{R}_+^N) \cap B_{r_0}(x_j^1) = \Omega \cap B_{r_0}(x_j^1), \qquad \Phi_j(\mathbb{R}_0^N) \cap B_{r_0}(x_j^1) = \Gamma \cap B_{r_0}(x_j^1).$$

3) There exist C^∞ functions ζ_j^i and $\tilde\zeta_j^i$ $(i = 0, 1, j \in N)$ such that $0 \le \zeta_j^i, \tilde\zeta_j^i \le 1$, $supp\,\zeta_j^i \subset supp\,\tilde\zeta_j^i \subset B_{r_0}(x_j^i)$, $\|\nabla\zeta_j^i\|_{H_\infty^{k-1}(\mathbb{R}^N)}, \|\nabla\tilde\zeta_j^i\|_{H_\infty^{k-1}(\mathbb{R}^N)} \le M_2$, $\tilde\zeta_j^i = 1$ on $supp\,\zeta_j^i$, $\sum_{i=0}^1 \sum_{j=1}^\infty \zeta_j^i = 1$ on $\overline{\Omega}$, $\sum_{j=1}^\infty \zeta_j^1 = 1$ on Γ.

4) There exists a natural number $L \ge 2$ such that any $L+1$ distinct sets of $\{B_{r_0}(x_j^i) \mid i = 0,1,2, j = 1,2,3,\ldots\}$ have an empty intersection.

Proof: A proof is given in Appendix 10. In the following, we use the symbols given in Proposition 6.1 and we write $\Omega_j = \Phi_j(\mathbb{R}_+^N)$, and $\Gamma_j = \Phi_j(\mathbb{R}_0^N)$ for the sake of simplicity. In view of the assumptions (1.2) and (1.3), we may assume that

$$|\mu(x) - \mu(x_j^0)| \le m_1 M_1 \quad \text{for any} x \in B_{r_0}(x_j^0);$$
$$|\mu(x) - \mu(x_j^1)| \le m_1 M_1, |\delta(x) - \delta(x_j^1)| \le m_1 M_1 \quad \text{for any } x \in \Omega_j \cap B_{r_0}(x_j^1);$$
$$|A_\sigma(x) - A_\sigma(x_j^1)| \le m_2 M_1 \quad \text{for any} x \in \Gamma_j \cap B_{r_0}(x_j^1); \tag{6.1}$$
$$m_0 \le \mu(x), \delta(x) \le m_1, \quad |\nabla\mu(x)|, |\nabla\delta(x)| \le m_1 \quad \text{for any } x \in \overline{\Omega},$$
$$|A_\sigma(x)| \le m_2 \quad \text{for any } x \in \Gamma, \quad \|A_\sigma\|_{W_r^{2-1/q}(\Gamma)} \le m_3\sigma^{-b} \quad \text{for any} \sigma \in (0,1). \tag{6.2}$$

Here, m_0, m_1, m_2, m_3, σ and r are constants given in (1.2) and (1.3).

We next prepare some propositions used to construct a parametrix. In the following, we write $B_j^i = B_{r_0}(x_j^i)$ for the sake of simplicity. By the finite intersection property stated in Proposition 6.1 iv, for any $r \in [1,\infty)$ there exists a constant $C_{r,L}$ such that

$$[\sum_{j=1}^\infty \|f\|_{L_r(\Omega \cap B_j^i)}^r]^{\frac{1}{r}} \le C_{r,L}\|f\|_{L_r(\Omega)} \quad \text{for any } f \in L_r(\Omega). \tag{6.3}$$

Proposition 6.2 Let X be a Banach space and X^* its dual space, while $\|\cdot\|_X$, $\|\cdot\|_{X^*}$, and $<\cdot,\cdot>$ be the norm of X, the norm of X^*, and the duality pairing between of X and X^*, respectively. Let $n \in \mathbb{N}$, $l = 1,\ldots,n$, and $\{a_l\}_{l=1}^n \subset C$, and let $\{f_j^l\}_{j=1}^\infty$ be sequences in X^* and $\{g_j^l\}_{j=1}^\infty$, $\{h_j\}_{j=1}^\infty$ be sequences of positive numbers. Assume that there exist maps $N_j : X \to [0,\infty)$ such that

$$|< f_j^l, \varphi >| \le M_3 g_j^l N_j(\varphi) \quad (l = 1,\ldots,n), \quad |< \sum_{l=1}^n a_l f_j^l, \varphi >| \le M_3 h_j N_j(\varphi)$$

for any $\varphi \in X$ with some positive constant M_3 independent of $j \in N$ and $l = 1,\ldots,n$. If

$$\sum_{j=1}^\infty (g_j^l)^q < \infty, \quad \sum_{j=1}^\infty (h_j)^q < \infty, \quad \sum_{j=1}^\infty (\eta(\phi))^{q'} \le (M_4\|\varphi\|_X)^{q'}$$

with $1 < q < \infty$ and $q' = q/(q-1)$ for some positive constant M_4, then the infinite sum $f^l = \sum_{j=1}^\infty f_j^l$ exists in the strong topology of X^* and

$$\|f^l\|_{X^*} \le M_3 M_4 (\sum_{j=1}^\infty (g_j^l)^q)^{1/q}, \quad \left\|\sum_{l=1}^n a_l f^l\right\|_{X^*} \le M_3 M_4 (\sum_{j=1}^\infty (h_j)^q)^{1/q}. \tag{6.4}$$

Proof: For a proof, see Proposition 9.5.2 in Shibata.[28] Let $n \in \mathbb{N}_0$, $f \in H_q^n(\Omega)$, and let η_j^i be functions in $C_0^\infty(B_j^i)$ with $\|\eta_j^i\|_{H_\infty^n(\mathbb{R}^N)} \le c_0$ for some constant c_0 independent of $j \in \mathbb{N}$. Since $\Omega \cap B_j^1 = \Omega_j \cap B_j^1$, by (6.3)

$$\sum_{j=1}^\infty \|\eta_j^0 f\|_{H_q^n(\mathbb{R}^N)}^q + \sum_{j=1}^\infty \|\eta_j^1 f\|_{H_q^n(\Omega_j)}^q \le C_q\|f\|_{H_q^n(\Omega)}^q. \tag{6.5}$$

The following propositions are used to define the infinite sum of R-bounded operator families defined on $\mathbb{R}^N$ and Ω_j.

Proposition 6.3 Let $1<q<\infty$, $i=0,1$, and $n\in\mathbb{N}_0$. Set $H_j^0=\mathbb{R}^N$ and $H_j^1=\Omega_j$. Let η_j^i be a function in $C_0^\infty(B_j^i)$ such that $\|\eta_j^i\|_{H_\infty^n(\mathbb{R}^N)}\le c_1$ for any $j\in\mathbb{N}$ with some constant c_1 independent of $j\in\mathbb{N}$. Let f_j $(j\in\mathbb{N})$ be elements in $H_q^n(H_j^i)$ such that $\sum_{j=1}^\infty\|f_j\|_{H_q^n(H_j^i)}^q<\infty$. Then, $\sum_{j=1}^\infty\eta_j^i f_j$ converges some $f\in H_q^n(\Omega)$ strongly in $H_q^n(\Omega)$, and

$$\|f\|_{H_q^n(\Omega)}\le C_q\{\sum_{j=1}^\infty\|f_j\|_{H_q^n(H_j^i)}^q\}^{1/q}.$$

Proof : For a proof, see Proposition 9.5.3 in Shibata.[35]

Proposition 6.4 Let $1<q<\infty$ and $n=2,3$. Then we have the following assertions.

1) There exist extension maps $T_j^n:W_q^{n-1/q}(\Gamma_j)\to H_q^n(\Omega_j)$ such that for any $h\in W_q^{n-1/q}(\Gamma_j)$, $T_j^n h=h$ on Γ_j and $\|T_j^n h\|_{H_q^n(\Omega_j)}\le C\|h\|_{W_q^{n-1/q}(\Gamma_j)}$ with some constant $C>0$ independent of $j\in\mathbb{N}$.
2) There exists an extension map $T_\Gamma^n:W_q^{n-1/q}(\Gamma)\to H_q^n(\Omega)$ such that for $h\in W_q^{n-1/q}(\Gamma)$, $T_\Gamma^n h=h$ on Γ and $\|T_\Gamma^n h\|_{H_q^n(\Omega)}\le C\|h\|_{W_q^{n-1/q}(\Gamma)}$ with some constant $C>0$.

Proof: For a proof, see Proposition 9.5.4 in Shibata.[35]

Proposition 6.5 Let $1<q<\infty$ and $n=2,3$ and let $\eta_j\in C_0^\infty(B_j^1)$ $(j\in N)$ with $\|\eta_j\|_{H_\infty^n(\mathbb{R}^N)}\le c_2$ for some constant c_2 independent of $j\in\mathbb{N}$. Then, we have the following two assertions:

1) Let f_j $(j\in\mathbb{N})$ be functions in $W_q^{n-1/q}(\Gamma_j)$ such that $\sum_{j=1}^\infty\|f_j\|_{W_q^{n-1/q}(\Gamma_j)}^q<\infty$, and then the infinite sum $\sum_{j=1}^\infty\eta_j f_j$ converges to some $f\in W_q^{n-1/q}(\Gamma)$ strongly in $W_q^{n-1/q}(\Gamma)$ and $\|f\|_{W_q^{n-1/q}(\Gamma)}\le C_q\{\sum_{j=1}^\infty\|f_j\|_{W_q^{n-1/q}(\Gamma_j)}^q\}^{1/q}$.
2) For any $h\in W_q^{n-1/q}(\Gamma)$, $\sum_{j=1}^\infty\|\eta_j h\|_{W_q^{n-1/q}(\Gamma_j)}^q\le C\|h\|_{W_q^{n-1/q}(\Gamma)}^q$.

Proof: For a proof, see Proposition 9.5.5 in Shibata.[35]

Parametrix

In this subsection, we construct a parametrix of reduced Stokes equations (2.4). Let $\{\zeta_j^i\}_{j\in\mathbb{N}}$ and $\{\tilde\zeta_j^i\}_{j\in\mathbb{N}}$ $(i=0,1)$ be sequences of C_0^∞ functions given in Proposition 6.1, and let $(f,d,h)\in Y_q(\Omega)$ (cf. (2.14)). Recall that $\Omega_j=\Phi_j(\mathbb{R}_+^N)$ and $\Gamma_j=\Phi_j(\mathbb{R}_0^N)$. Let

$$\mu_j^i(x)=\tilde\zeta_j^i(x)\mu(x)+(1-\tilde\zeta_j^i(x))\mu(x_j^i),\quad \delta_j(x)=\tilde\zeta_j^1(x)\delta(x)+(1-\tilde\zeta_j^1(x))\delta(x_j^1),$$
$$A_{\sigma,j}(x)=\tilde\zeta_j^1(x)A_\sigma(x)+(1-\tilde\zeta_j^1(x))A_\sigma(x_j^1).$$

Notice that

$$\zeta_j^i\mu=\zeta_j^i\mu_j^i,\quad \zeta_j^1\delta=\zeta_j^1\delta_j,\quad \zeta_j^1A_\sigma=\zeta_j^1A_{\sigma,j},$$

because $\tilde\zeta_j^1=1$ on $supp\,\zeta_j^1$. We consider the equations:

$$\lambda u_j^0-\mathrm{Div}(\mu_j^0D(u_j^0)-K_{0j}(u_j^0)I)=\tilde\zeta_j^0 f\qquad\text{in }\mathbb{R}^N; \tag{6.6}$$

$$\begin{cases}\lambda u_j^1-\mathrm{Div}(\mu_j^1D(u_j^1)-K_{1j}(u_j^1,h_j)I) & =\tilde\zeta_j^1 f & \text{in }\Omega_j,\\ \lambda h_j+A_{\sigma,j}\cdot\nabla_{\Gamma_j}h_j-n_j\cdot u_j & =\tilde\zeta_j^1 d & \text{on }\Gamma_j,\\ (\mu_j^1D(u_j^1)-K_{1j}(u_j^1,h_j)I)n_j-\delta_j(\Delta_{\Gamma_j}h_j)n_j & =\tilde\zeta_j^1 h & \text{on }\Gamma_j.\end{cases} \tag{6.7}$$

Here, for $u\in H_q^2(\mathbb{R}^N)^N$, $K_{0j}(u)\in\hat H_q^1(\mathbb{R}^N)$ denotes a unique solution of the weak Laplace equation:

$$(\nabla K_{0j}(u),\nabla\phi)_{\mathbb{R}^N}=(\mathrm{Div}(\mu_j^0D(u))-\nabla\mathrm{div}u,\nabla\phi)_{\mathbb{R}^N}\quad\text{for any}\,\phi\in\hat{\mathrm{H}}_{q'}^1(\mathbb{R}^N). \tag{6.8}$$

And, for $u\in H_q^2(\Omega_j)$ and $h\in H_q^3(\Omega_j)$, $K_{1j}(u,h_j)\in H_q^1(\Omega_j)+\hat H_{q,0}^1(\Omega_j)$ denotes a unique solution of the weak Dirichlet problem:

$$(\nabla K_{1j}(u,h),\nabla\varphi)_{\Omega_j}=(\mathrm{Div}(\mu_j^1D(u))-\nabla\mathrm{div}u,\nabla\varphi)_{\Omega_j}\quad\text{for any}\,\varphi\in\hat{\mathrm{H}}_{q',0}^1(\Omega_j), \tag{6.9}$$

subject to $K_{1j}(u,h)=<\mu_j^1D(u)n_j,n_j>-\mathrm{div}u-\delta_j\Delta_{\Gamma_j}h$ on Γ_j. Moreover, we denote the unit outer normal to Γ_j by n_j, which are defined on $\mathbb{R}^N$ and satisfies the estimate:

$$\|n_j\|_{L_\infty(\mathbb{R}^N)}\le C,\quad \|\nabla n_j\|_{L_\infty(\mathbb{R}^N)}\le C_K,\quad \|\nabla^2n_j\|_{L_\infty(\mathbb{R}^N)}\le C_{M2}.$$

Let $\nabla_{\Gamma_j} = (\partial_1, \ldots, \partial_{N-1})$ with $\partial_j = \partial / \partial x_j$ for $y = \Phi_j(x', 0) \in \Gamma_j$ and let Δ_{Γ_j} be the Laplace-Beltrami operator on Γ_j, which have the form:

$$\Delta_{\Gamma_j} f = \Delta' f + D_{\Gamma_j} f \quad \text{on } \Phi_j^{-1}(\Gamma_j),$$

where $\Delta' f = \sum_{j=1}^{N-1} \partial_j^2 f$ and $D_{\Gamma_j} f = \sum_{k,\ell=1}^{N-1} a^j_{k\ell} \partial_k \partial_\ell f + \sum_{k=1}^{N-1} a^j_k \partial_k f$, and $a^j_{k\ell}$ and a^j_k satisfy the following estimates:

$$\|a^j_{k\ell}\|_{L_\infty(\mathbb{R}^N)} \le CM_1, \quad \|(\partial_1 a^j_{k\ell}, \ldots, \partial_{N-1} a^j_{k\ell}, a^j_k)\|_{L_\infty(\mathbb{R}^N)} \le C_K,$$
$$\|(\partial_1 a^j_{k\ell}, \ldots, \partial_{N-1} a^j_{k\ell}, a^j_k)\|_{H^1_\infty(\mathbb{R}^N)} \le C_{M_2}.$$

Notice that $n_j = n$ and $\Delta_{\Gamma_j} = \Delta_\Gamma$ on $\Gamma_j \cap B^1_j = \Gamma \cap B^1_j$. We know the existence of $K_{0j}(u^0_j) \in \hat{H}^1_q(\mathbb{R}^N)$ possessing the estimate:

$$\|\nabla K_{0j}(u^0_j)\|_{L_q(\mathbb{R}^N)} \le C \|\nabla u^0_j\|_{H^1_q(\mathbb{R}^N)}. \tag{6.10}$$

Let ρ be a function in $C_0^\infty(B_{r_0})$ such that $\int_{R^N} \rho dx = 1$. Below, this ρ is fixed. Since $K_{0j}(u^0_j) + c$ also satisfy the variational equation (6.8) for any constant c, we may assume that

$$\int_{B^0_j} K_{0j}(u^0_j) \rho(x - x^0_j) dx = 0. \tag{6.11}$$

Moreover, choosing $M_1 \in (0,1)$ suitably small, we have the unique existence of solutions $K_{1j}(u^1_j, h_j) \in H^1_q(\Omega_j) + \hat{H}^1_{q,0}(\Omega_j)$ of Eq.(6.9) possessing the estimates:

$$\|\nabla K_{1j}(u^1_j, h_j)\|_{L_q(\Omega_j)} \le C(\|\nabla u^1_j\|_{H^1_q(\Omega_j)} + \|h_j\|_{W_q^{3-1/q}(\Gamma_j)}). \tag{6.12}$$

Let $Y_q(\Omega_j)$ and $Y_q(\Omega_j)$ be the spaces defined in (2.14) replacing Ω by Ω_j. By Theorem 3.1 and Theorem 5.1, there exist constants $M_1 \in (0,1)$ and $\lambda_0 \ge 1$, which are independent of $j \in \mathbb{N}$, and operator families

$$S_{0j}(\lambda) \in \mathrm{Hol}(\Sigma_{\varepsilon,\lambda_0}, L(L_q(\mathbb{R}^N)^N, H^2_q(\mathbb{R}^N)^N)), \quad S_{1j}(\lambda) \in \mathrm{Hol}(\Lambda_{\sigma,\lambda_0\gamma_\sigma}, L(Y_q(\Omega_j), H^2_q(\Omega_j)^N)),$$
$$H_j(\lambda) \in \mathrm{Hol}(\Lambda_{\sigma,\lambda_0\gamma_\sigma}, L(Y_q(\Omega_j), H^3_q(\Omega_j)))$$

such that for each $j \in \mathbb{N}$, Eq.(6.6) admits a unique solution $u^0_j = S_{0j}(\lambda)\tilde{\zeta}^0_j f$ and Eq.(6.7) admits unique solutions $u^1_j = S_{1j}(\lambda)\tilde{\zeta}^1_j F_\lambda(f,d,h)$ and $h_j = H_j(\lambda)\tilde{\zeta}^1_j F_\lambda(f,d,h)$, where $F_\lambda(f,d,h) = (f, d, \lambda^{1/2}h, h)$, and $\tilde{\zeta}^1_j F_\lambda(f,d,h) = (\tilde{\zeta}^1_j f, \tilde{\zeta}^1_j d, \lambda^{1/2}\tilde{\zeta}^1_j h, \tilde{\zeta}^1_j h)$. Moreover, there exists a number $r_b > 0$ independent of M_1, M_2, and $j \in \mathbb{N}$ such that

$$\begin{aligned}
&R_{L(L_q(\mathbb{R}^N)^N, H_q^{2-k}(\mathbb{R}^N)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2} S_{0j}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) &\le r_b,\\
&R_{L(Y_q(\Omega_j), H_q^{2-k}(\Omega_j)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{k/2} S_{1j}(\lambda)) \mid \lambda \in \Lambda_{\varepsilon,\lambda_0\gamma_\sigma}\}) &\le r_b,\\
&R_{L(Y_q(\Omega_j), H_q^{3-n}(\Omega_j))}(\{(\tau\partial_\tau)^\ell(\lambda^n H_j(\lambda)) \mid \lambda \in \Lambda_{\sigma,\lambda_0\gamma_\sigma}\}) &\le r_b,
\end{aligned} \tag{6.13}$$

for $\ell = 0,1$, $j \in N$, $k = 0,1,2$, and $n = 0,1$. Notice that $\lambda_0\gamma_\sigma \ge \lambda_0$.

By (6.13), we have

$$\begin{aligned}
&|\lambda| \|u^0_j\| u^0_{j\,L_q(\mathbb{R}^N)} + |\lambda|^{1/2} \|u^0_j\|_{H^1_q(\mathbb{R}^N)} + \|u^0_j\|_{H^2_q(\mathbb{R}^N)} \le r_b \|\zeta^0_j f\|_{L_q(\mathbb{R}^N)},\\
&|\lambda| \|u^1_j\|_{L_q(\Omega_j)} + |\lambda|^{1/2} \|u^1_j\|_{H^1_q(\Omega_j)} + \|u^1_j\|_{H^2_q(\Omega_j)} + |\lambda| \|h_j\|_{H^2_q(\Omega_j)} + \|h_j\|_{H^3_q(\Omega_j)}\\
&\quad \le r_b(\|\tilde{\zeta}^1_j f\|_{L_q(\Omega_j)} + \|\tilde{\zeta}^1_j d\|_{W_q^{2-1/q}(\Gamma_j)} + |\lambda|^{1/2} \|h\|_{L_q(\Omega_j)} + \|h\|_{H^1_q(\Omega_j)})
\end{aligned} \tag{6.14}$$

for $\lambda \in \Sigma_{\sigma,\lambda_0\gamma_\sigma}$. Let

$$u = \sum_{i=0}^{1}\sum_{j=1}^{\infty} \zeta^i_j u^i_j, \quad h = \sum_{j=1}^{\infty} \zeta^1_j h_j. \tag{6.15}$$

Then, by (6.6), (6.7), (6.14), Proposition 6.3, and Proposition 6.5, we have $u \in H^2_q(\Omega)^N$, $h \in H^3_q(\Omega)$, and

$$|\lambda| \|u\|_{L_q(\Omega)} + |\lambda|^{1/2} \|u\|_{H^1_q(\Omega)} + \|u\|_{H^2_q(\Omega)} + |\lambda| \|h\|_{H^2_q(\Omega)} + \|h\|_{H^3_q(\Omega)}$$
$$\le C_q r_b(\|f\|_{L_q(\Omega)} + \|d\|_{W_q^{2-1/q}(\Gamma)} + |\lambda|^{1/2} \|h\|_{L_q(\Omega)} + \|h\|_{H^1_q(\Omega)}\}$$

for $\lambda \in \Lambda_{\sigma,\lambda_0}$.

Moreover, we have

$$\begin{cases} \lambda u - \mathrm{Div}(\mu \mathrm{D}(\mathrm{u}) - K(\mathrm{u},h)I) & = f - V^1(\lambda)(f,d,\mathrm{h}) \quad \text{in } \Omega, \\ \lambda h + A_\sigma \cdot \nabla_\Gamma h - \mathrm{u}\cdot\mathrm{n} + \mathcal{F}\,\mathrm{u} & = d - V_\sigma^2(\lambda)(f,d,\mathrm{h}) \quad \text{on } \Gamma, \\ (\mu \mathrm{D}(\mathrm{u}) - K(\mathrm{u},h)I - ((\mathcal{B} + \delta\Delta_\Gamma)h)\mathrm{I})\mathrm{n} & = h - V^3(\lambda)(f,d,\mathrm{h}) \quad \text{on } \Gamma, \end{cases} \tag{6.16}$$

where we have set

$$V^1(\lambda)(f,d,h) = V_1^1(\lambda)(f,d,h) + V_2^1(\lambda)(f,d,h),$$

$$V_1^1(\lambda)(f,d,h) = \sum_{i=0}^{1}\sum_{j=1}^{\infty}[\mathrm{Div}(\mu(D(\zeta_j^i u_j^i) - \zeta_j^i D(u_j^i))) + \mathrm{Div}(\zeta_j^i \mu_j^i D(u_j^i)) - \zeta_j^i \mathrm{Div}(\mu_j^i D(u_j^i))],$$

$$V_2^1(\lambda)(f,d,h) = \nabla K(u,h) - \sum_{j=1}^{\infty}\zeta_j^0 \nabla K_{0j}(u_j^0) - \sum_{j=1}^{\infty}\zeta_j^1 \nabla K_{1j}(u_j^1,h_j),$$

$$V_\sigma^2(\lambda)(f,d,h) = \sum_{j=1}^{\infty} A_\sigma(x)\cdot((\nabla_\Gamma \zeta_j^1)h_j) - \sum_{j=1}^{\infty}\mathcal{F}\,(\zeta_j^1 u_j^1),$$

$$V^3(\lambda)(f,d,h) = V_1^3(\lambda)(f,d,h) - V_2^3(\lambda)(f,d,h) - V_3^3(\lambda)(f,d,h),$$

$$V_1^3(\lambda)(f,d,h) = \sum_{j=1}^{\infty}\mu(D(\zeta_j^1 u_j^1) - \zeta_j^1 D(u_j^1))n, \quad V_2^3(\lambda)(f,d,h) = \{\sum_{j=1}^{\infty}\zeta_j^1 K_{1j}(u_j^1,h_j) - K(u,h)\}n,$$

$$V_3^3(\lambda)(f,d,h) = \sum_{j=1}^{\infty}\{\delta(\Delta_{\Gamma_j}(\zeta_j^1 h_j^1) - \zeta_j^1 \Delta_{\Gamma_j} h_j^1) + \mathcal{B}(\zeta_j^1 h_j^1)\}.$$

For $F = (F_1, F_2, F_3, F_4) \in Y_q(\Omega)$, we define operators $\mathcal{A}_p(\lambda)$ and $\mathcal{B}_p(\lambda)$ acting on F by

$$\mathcal{A}_p(\lambda)F = \sum_{j=1}^{\infty}\zeta_j^0 \mathcal{S}_j^0(\lambda)\tilde\zeta_j^0 F_1 + \sum_{j=1}^{\infty}\zeta_j^1 \mathcal{S}_{1j}(\lambda)\tilde\zeta_j^1 F, \quad \mathcal{B}_p(\lambda)F = \sum_{j=1}^{\infty}\zeta_j^1 \mathcal{H}_j(\lambda)\tilde\zeta_j^1 F. \tag{6.17}$$

Then, by Proposition 6.3 and (6.13), we have $u = \mathcal{A}_p(\lambda)F_\lambda(f,d,h)$, $h = \mathcal{H}_p(\lambda)F_\lambda(f,d,h)$, and

$$\begin{aligned} &\mathcal{A}_p(\lambda) \in \mathrm{Hol}(\Lambda_{\sigma,\lambda_1}, \mathcal{L}(Y_q(\Omega), H_q^2(\Omega)^N)), \quad \mathcal{B}_p(\lambda) \in \mathrm{Hol}(\Lambda_{\sigma,\lambda_1}, \mathcal{L}(Y_q(\Omega), H_q^3(\Omega))), \\ &\mathcal{R}_{\mathcal{L}(Y_q(\Omega), H_q^{2-j}(\Omega)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}\mathcal{A}_p(\lambda)) \mid \lambda \in \Lambda_{\sigma,\lambda_1}\}) \le (C + C_{M_2}\lambda_1^{-1/2})r_b, \\ &\mathcal{R}_{\mathcal{L}(Y_q(\Omega), H_q^{3-k}(\Omega))}(\{(\tau\partial_\tau)^\ell(\lambda^k \mathcal{B}_p(\lambda)) \mid \lambda \in \Lambda_{\sigma,\lambda_1}\}) \le (C + C_{M_2}\lambda_1^{-1})r_b \end{aligned} \tag{6.18}$$

for $\ell = 0,1$, $j = 0,1,2$, and $k = 0,1$ for any $\lambda_1 \ge \lambda_0\gamma_\sigma$.

Estimates of the remainder terms

For $F = (F_1, F_2, F_3, F_4) \in Y_q(\Omega)$, let

$$V^1(\lambda)F = V_1^1(\lambda)F + V_2^1(\lambda)F,$$

$$\begin{aligned} V_1^1(\lambda)F = &\sum_{j=1}^{\infty}[\mathrm{Div}(\mu(\mathrm{D}(\zeta_j^0 \mathcal{S}_{0j}(\lambda)\tilde\zeta_j^0 F_1) - \zeta_j^0 \mathrm{D}(\mathcal{S}_{0j}(\lambda)\tilde\zeta_j^0 F_1))) \\ &+ \mathrm{Div}(\zeta_j^0 \mu_j^0 \mathrm{D}(\mathcal{S}_{0j}(\lambda)\tilde\zeta_j^0 F_1)) - \zeta_j^0 \mathrm{Div}(\mu_j^0 \mathrm{D}(\mathcal{S}_{0j}(\lambda)\tilde\zeta_j^0 F_1))] \\ &+ \sum_{j=1}^{\infty}[\mathrm{Div}(\mu(\mathrm{D}(\zeta_j^1 \mathcal{S}_{1j}(\lambda)\tilde\zeta_j^1 F) - \zeta_j^1 \mathrm{D}(\mathcal{S}_{1j}(\lambda)\tilde\zeta_j^1 F))) \\ &+ \mathrm{Div}(\zeta_j^1 \mu_j^1 \mathrm{D}(\mathcal{S}_{1j}(\lambda)\tilde\zeta_j^1 F)) - \zeta_j^1 \mathrm{Div}(\mu_j^1 \mathrm{D}(\mathcal{S}_{1j}(\lambda)\tilde\zeta_j^1 F))], \end{aligned}$$

$$V_2^1(\lambda)F = \nabla K(\mathcal{A}_p(\lambda)F, \mathcal{B}_p(\lambda)F) - \sum_{j=1}^{\infty}\zeta_j^0 \nabla K_{0j}(\mathcal{S}_{0j}(\lambda)\tilde\zeta_j^0 F_1) - \sum_{j=1}^{\infty}\zeta_j^1 \nabla K_{1j}(\mathcal{S}_{1j}(\lambda)\tilde\zeta_j^1 F, \mathcal{H}_j(\lambda)\tilde\zeta_j^1 F),$$

$$V_\sigma^2(\lambda)F = \sum_{j=1}^{\infty} A_\sigma(x)\cdot((\nabla_\Gamma \zeta_j^1)\mathcal{H}_j(\lambda)\tilde\zeta_j^1 F) - \sum_{j=1}^{\infty}\mathcal{F}\,(\zeta_j^1 \mathcal{S}_{1j}(\lambda)\tilde\zeta_j^1 F),$$

$$V^3(\lambda)F = V_1^3(\lambda)F + V_2^3(\lambda)F + V_3^3(\lambda)F,$$

$$V_1^3(\lambda)F = \sum_{j=1}^{\infty}\mu(D(\zeta_j^1 \mathcal{S}_{1j}(\lambda)\tilde\zeta_j^1 F) - \zeta_j^1 \mathrm{D}(\zeta_j^1 \mathcal{S}_{1j}(\lambda)\tilde\zeta_j^1 F))\mathrm{n}$$

$$V_2^3(\lambda)F = \{\sum_{j=1}^{\infty}\zeta_j^1 K_{1j}(S_{1j}(\lambda)\tilde{\zeta}_j^1 F, H_j(\lambda)\tilde{\zeta}_j^1 F) - K(A_p(\lambda)F, B_p(\lambda)F)\}\mathrm{n},$$

$$V_3^3(\lambda)F = \sum_{j=1}^{\infty}\{\delta(\Delta_{\Gamma_j}(\zeta_j^1 H_j(\lambda)\tilde{\zeta}_j^1 F) - \zeta_j^1\Delta_{\Gamma_j}H_j(\lambda)\tilde{\zeta}_j^1 F) + B(\zeta_j^1 H_j(\lambda)\tilde{\zeta}_j^1 F)\}$$

Notice that $V_\sigma^2(\lambda)F = \sum_{j=1}^{\infty} F\,(\zeta_j^1 S_{1j}(\lambda)\tilde{\zeta}_j^1 F)$ for $\sigma = 0$.

Let

$$V(\lambda)(f,d,h) = (V^1(\lambda)(f,d,h), V^2(\lambda)(f,d,h), V^3(\lambda)(f,d,h)), \quad V(\lambda)F = (V^1(\lambda)F, V_\sigma^2(\lambda)F, V^3(\lambda)F).$$

Since $u_j^0 = S_{0j}(\lambda)\zeta_j^0 f$, $\mathrm{u}_j^1 = S_{1j}(\lambda)\tilde{\zeta}_j^1 \mathrm{F}_\lambda(f,d,h)$, and $h_j = H_j(\lambda)\zeta_j^1 \mathrm{F}_\lambda\tilde{(}f,d,\mathrm{h})$, we have

$$V(\lambda)(f,d,\mathrm{h}) = V(\lambda)F_\lambda(f,d,\mathrm{h}). \tag{6.19}$$

In what follows, we shall prove that

$$R_{L(Y_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\mathrm{F}_\lambda V(\lambda)) \mid \lambda \in \Lambda_{\sigma,\tilde{\lambda}_0}\}) \le C_q r_b(\varepsilon + C_{M_2,\varepsilon}(\tilde{\lambda}_0^{-1}\gamma_\sigma + \tilde{\lambda}_0^{-1/2})) \tag{6.20}$$

for $\ell = 0,1$ and $\tilde{\lambda}_0 \ge \lambda_0$, where γ_σ is the number given in Theorem 1.7.

To prove (6.20), we use Proposition 6.1, Proposition 3.4, Propositions 6.2–6.5, (5.6), (5.7), (6.1), (6.2), (6.5) and (6.13). In the following, $\tilde{\lambda}_0$ is any number such that $\tilde{\lambda}_0 \ge \lambda_0$. We start with the following estimate of $V_1^1(\lambda)$:

$$R_{L(Y_q(\Omega), L_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell V_1^1(\lambda) \mid \lambda \in \Sigma_{\sigma,\tilde{\lambda}_0}\}) \le C_{M_2} r_b \tilde{\lambda}_0^{-1/2} \quad (\ell = 0,1). \tag{6.21}$$

In fact, since $D_{\ell,m}(\zeta_j^i u) - \zeta_j^i D_{\ell m}(u) = (\partial_\ell \zeta_j^i)u_m + (\partial_m \zeta_j^i)u_\ell$, and $\mathrm{div}(\zeta_j^i u) - \zeta_j^i \mathrm{div} u = \sum_{k=1}^N (\partial_k \zeta_j^i)u_k$, for any $n \in \mathbb{N}$, $\{\lambda_\ell\}_{\ell=1}^n \subset \Lambda_{\sigma,\tilde{\lambda}_0}^n$, and $\{F_\ell = (F_{1\ell}, F_{2\ell}, F_{3\ell}, F_{4\ell})\}_{\ell=1}^n \subset Y_q(\Omega)^n$, we have

$$\int_0^1 \left\| \sum_{\ell=1}^n r_\ell(u) V_1^1(\lambda_\ell) F_\ell \right\|_{L_q(\Omega)}^q du$$

$$\le C_q^q M_2^q \sum_{j=1}^{\infty}\{\int_0^1 \left\| \sum_{\ell=1}^n r_\ell(u) S_{0j}(\lambda_\ell)\tilde{\zeta}_j^0 F_{1\ell} \right\|_{H_q^1(\mathbb{R}^N)}^q du + \int_0^1 \left\| \sum_{\ell=1}^n r_\ell(u) S_{1j}(\lambda_\ell)\tilde{\zeta}_j^1 F_\ell \right\|_{H_q^1(\Omega_j)}^q du\}$$

$$\le C_q^q M_2^q \tilde{\lambda}_0^{-q/2} \sum_{j=1}^{\infty}\{\int_0^1 \left\| \sum_{\ell=1}^n r_\ell(u) \lambda_\ell^{1/2} S_{0j}(\lambda_\ell)\tilde{\zeta}_j^0 F_{1\ell} \right\|_{H_q^1(\mathbb{R}^N)}^q du$$

$$+ \int_0^1 \left\| \sum_{\ell=1}^n r_\ell(u) \lambda_\ell^{1/2} S_{1j}(\lambda_\ell)\tilde{\zeta}_j^1 F_\ell \right\|_{H_q^1(\Omega_j)}^q du\}$$

$$\le C_q^q M_2^q \tilde{\lambda}_0^{-q/2} r_b^q \sum_{j=1}^{\infty}\{\int_0^1 \left\| \sum_{\ell=1}^n r_\ell(u) \tilde{\zeta}_j^0 F_{1\ell} \right\|_{L_q(\mathbb{R}^N)}^q du + \int_0^1 \left\| \sum_{\ell=1}^n r_\ell(u) \tilde{\zeta}_j^1 F_\ell \right\|_{Y_q(\Omega_j)}^q du\}$$

$$\le C_q^{2q} M_2^q \tilde{\lambda}_0^{-q/2} r_b^q \int_0^1 \left\| \sum_{\ell=1}^n r_\ell(u) F_\ell \right\|_{L_q(\Omega)}^q du.$$

This shows that

$$R_{L(Y_q(\Omega), L_q(\Omega)^N)}(\{V_1^1(\lambda) \mid \lambda \in \Sigma_{\sigma,\tilde{\lambda}_0}\}) \le C_{M_2} r_b \tilde{\lambda}_0^{-1/2}.$$

Analogously, we can show that

$$R_{L(Y_q(\Omega), L_q(\Omega)^N)}(\{\tau\partial_\tau V_1^1(\lambda) \mid \lambda \in \Sigma_{\sigma,\tilde{\lambda}_0}\}) \le C_{M_2} r_b \tilde{\lambda}_0^{-1/2},$$

and therefore we have (6.21).

For $r \in (N,\infty)$ and $q \in (1,\infty)$, by the extension of functions defined on Γ_j to Ω_j and Sobolev's imbedding theorem, we have

$$\|ab\|_{W_q^{2-1/q}(\Gamma_j)} \le C_{q,r,K} \|a\|_{H_q^2(\Omega_j)} \|b\|_{W_q^{2-1/q}(\Gamma_j)}$$

for any $a \in H_r^2(\Omega)$ and $b \in W_q^{2-1/q}(\Gamma_j)$. Applying this inequality, we have

$$\|A_\sigma \cdot ((\nabla_\Gamma \zeta_j^1) H_j(\lambda)\tilde{\zeta}_j^1 F)\|_{W_q^{2-1/q}(\Gamma_j)} \le Cq,rM_2 m_3 \sigma^{-b} \|H_j(\lambda)\tilde{\zeta}_j^1 F\|_{H_q^2(\Omega_j)}$$

for $\sigma \in (0,1)$. Thus, employing the same argument as in the proof of (6.21), we have

$$R_{L(Y_q(\Omega), W_q^{2-1/q}(\Gamma))}(\{(\tau\partial_\tau)^\ell V_0^2(\lambda) \mid \lambda \in \Lambda_{\sigma,\tilde\lambda_0}\}) \le C_{M_2} r_b \tilde\lambda_0^{-1/2} \quad (\ell = 0,1),$$

$$R_{L(Y_q(\Omega), W_q^{2-1/q}(\Gamma))}(\{(\tau\partial_\tau)^\ell V_\sigma^2(\lambda) \mid \lambda \in \Lambda_{\sigma,\tilde\lambda_0}\}) \le C_{M_2} r_b (\tilde\lambda_0^{-1}\sigma^{-b} + \tilde\lambda_0^{-1/2}) \quad (\ell = 0,1)$$

for $\sigma \in (0,1)$.

Employing the same argument as in the proof of (6.21), we also have

$$R_{L(Y_q(\Omega), L_q(\Gamma)^N)}(\{(\tau\partial_\tau)^\ell (\lambda^{1/2} V_m^3(\lambda)) \mid \lambda \in \Lambda_{\sigma,\tilde\lambda_0}\}) \le C_{M_2} r_b \tilde\lambda_0^{-1/2} \quad (\ell = 0,1),$$

$$R_{L(Y_q(\Omega), H_q^1(\Gamma)^N)}(\{(\tau\partial_\tau)^\ell V_m^3(\lambda) \mid \lambda \in \Lambda_{\sigma,\tilde\lambda_0}\}) \le C_{M_2} r_b \tilde\lambda_0^{-1/2} \quad (\ell = 0,1),$$

for $m = 1$ and 3. Noting that $\mu\zeta_j^1 = \mu_j^1\zeta_j^1$ and $\delta\zeta_j^1 = \delta_j^1\zeta_j^1$, we have

$$\begin{aligned}
&\sum_{j=1}^\infty \zeta_j^1 K_{1j}(S_{1j}(\lambda)\tilde\zeta_j^1 F, H_j(\lambda)\tilde\zeta_j^1 F) - K(A_p(\lambda)F, B_p(\lambda)F) \\
&= \sum_{j=1}^\infty \mu < \zeta_j^1 \mathrm{D}(S_{1j}(\lambda)\tilde\zeta_j^1 F) - \mathrm{D}(\zeta_j^1 S_{1j}(\lambda)\tilde\zeta_j^1 F), n > \\
&- \sum_{j=1}^\infty \{\zeta_j^1 \mathrm{div} S_{1j}(\lambda)\tilde\zeta_j^1 F - \mathrm{div}(\zeta_j^1 S_{1j}(\lambda)\tilde\zeta_j^1 F)\} \\
&- \sum_{j=1}^\infty \delta\{\zeta_j^1 \Delta_{\Gamma_j}(H_j(\lambda)\tilde\zeta_j^1 F) - \Delta_{\Gamma_j}(\zeta_j^1 H_j(\lambda)\tilde\zeta_j^1 F)\} - \sum_{j=1}^\infty B(\zeta_j^1 H_j(\lambda)\tilde\zeta_j^1 F)
\end{aligned} \qquad (6.22)$$

on Γ, where we have used $\Delta_\Gamma = \Delta_{\Gamma_j}$ and $\mathrm{n} = \mathrm{n}_j$ on $\Gamma_j \cap B_j^1$. Employing the same argument as in the proof of (6.21), we have

$$R_{L(Y_q(\Omega), L_q(\Gamma)^N)}(\{(\tau\partial_\tau)^\ell (\lambda^{1/2} V_2^3(\lambda)) \mid \lambda \in \Lambda_{\sigma,\tilde\lambda_0}\}) \le C_{M_2} r_b \tilde\lambda_0^{-1/2} \quad (\ell = 0,1),$$

$$H_{L(Y_q(\Omega), H_q^1(\Gamma)^N)}(\{(\tau\partial_\tau)^\ell V_2^3(\lambda) \mid \lambda \in \Lambda_{\sigma,\tilde\lambda_0}\}) \le C_{M_2} r_b \tilde\lambda_0^{-1/2} \quad (\ell = 0,1).$$

The final task is to prove that

$$R_{L(Y_q(\Omega), L_q(\Gamma)^N)}(\{(\tau\partial_\tau)^\ell V_2^1(\lambda) \mid \lambda \in \Lambda_{\sigma,\tilde\lambda_0}\}) \le C_{q,r}(\varepsilon + C_{M_2,\varepsilon} \tilde\lambda_0^{-1/2}) r_b \quad (\ell = 0,1). \qquad (6.23)$$

For this purpose, we use the following lemmata.

Lemma 6.6: Let Ω be a uniformly C^2 domain in $\mathbb{R}^N$. Then, there exists a constant $c_1 > 0$ independent of $j \in \mathbb{N}$ such that

$$\begin{aligned}
2\|\varphi\|_{H_q^1(\Omega_j \cap B_j^1)} &\le c_1 \|\nabla\varphi\|_{L_q(\Omega_j \cap B_j^1)} \quad \text{for any} \varphi \in \hat{\mathrm{H}}_{\mathrm{q},0}^1(\Omega_j), \\
\|\psi\|_{H_q^1(\Omega \cap B_j^1)} &\le c_1 \|\nabla\psi\|_{L_q(\Omega \cap B_j^1)} \quad \text{for any} \psi \in \hat{\mathrm{H}}_{\mathrm{q},0}^1(\Omega), \\
\|\varphi - c_j(\varphi)\|_{H_q^1(B_j^0)} &\le c_1 \|\nabla\varphi\|_{L_q(B_j^0)} \quad \text{for any} \varphi \in \hat{\mathrm{H}}_{\mathrm{q}}^1(\mathbb{R}^{\mathrm{N}}), \\
\|\psi - c_j(\psi)\|_{H_q^1(B_j^0)} &\le c_1 \|\nabla\psi\|_{L_q(B_j^0)} \quad \text{for any} \psi \in \hat{\mathrm{H}}_{\mathrm{q},0}^1(\Omega).
\end{aligned}$$

Here, $c_j(\varphi)$ and $c_j(\psi)$ are suitable constants depending on φ and ψ, respectively.

Proof: For a proof, see Shibata [Lemma 3.4, Lemma 3.5].[43]

Lemma 6.7 Let $1 < q < \infty$. For $u \in H_q^2(\mathbb{R}^N)$, let $K_{0j}(u)$ be a unique solution of the weak Laplace equation (6.8) satisfying (6.11). Then, we have

$$\|K_{0j}(u)\|_{L_q(B_j^0)} \le C\|\nabla u\|_{L_q(\mathbb{R}^N)}. \qquad (6.24)$$

Proof: Let ρ be the same function in (6.11). Let ψ be any function in $C_0^\infty(B_j^0)$ and we set

$$\tilde\psi(x) = \psi(x) - \rho(x - x_j^0)\int_{\mathbb{R}^N} \psi(y)\,dy.$$

Then,

$$\tilde\psi \in C_0^\infty(B_j^0), \quad \int_{\mathbb{R}^N} \tilde\psi\,dx = 0, \quad \|\tilde\psi\|_{L_{q'}(B_j^0)} \le C_{q'} \|\psi\|_{q'(B_j^0)}. \qquad (6.25)$$

Moreover,

$$\tilde{\psi} \in \hat{H}_q^1(\mathbb{R}^N)^* = \hat{H}_{q'}^{-1}(\mathbb{R}^N), \quad \|\tilde{\psi}\|_{\hat{H}_{q'}^{-1}(\mathbb{R}^N)} \le C_{q'} \|\psi\|_{L_{q'}(\mathbb{R}^N)}. \tag{6.26}$$

In fact, by Lemma 6.6, for any $\varphi \in \hat{H}_{q'}^1(\mathbb{R}^N)$, there exists a constant e_j for which

$$\|\varphi - e_j\|_{L_q(B_j^0)} \le c_q \|\nabla \varphi\|_{L_q(B_j^0)}.$$

Thus, by (6.25), we have

$$|(\tilde{\psi},\varphi)_{\mathbb{R}^N}| = |(\tilde{\psi},\varphi - e_j)_{\mathbb{R}^N}| \le \|\tilde{\psi}\|_{L_{q'}(B_j^0)} \|\varphi - e_j\|_{L_q(B_j^0)} \le C_q \|\tilde{\psi}\|_{q'(B_j^0)} \|\nabla\varphi\|_{L_q(B_j^0)},$$

which yields (6.26). Let Ψ be a function in $\hat{H}_{q'}^1(\mathbb{R}^N)$ such that $\nabla\Psi \in H_{q'}^1(\mathbb{R}^N)^N$,

$$(\nabla\Psi,\nabla\theta)_{\mathbb{R}^N} = (\tilde{\psi},\theta)_{\mathbb{R}^N} \quad \text{for any } \theta \in \hat{\mathrm{H}}_{\mathrm{q}}^1(\mathbb{R}^{\mathrm{N}}), \quad \|\nabla\Psi\|_{H_{q'}^1(\mathbb{R}^N)} \le C(\|\tilde{\psi}\|_{L_{q'}(\mathbb{R}^N)} + \|\tilde{\psi}\|_{\hat{H}_{q'}^{-1}(\mathbb{R}^N)}). \tag{6.27}$$

By (6.25) and (6.26), we have

$$\|\nabla\Psi\|_{H_{q'}^1(\mathbb{R}^N)} \le C_{q'} \|\psi\|_{L_{q'}(\mathbb{R}^N)}. \tag{6.28}$$

By (6.11), (6.27), and the divergence theorem of Gauss, we have

$$\begin{aligned}(K_{0j}(u),\psi)_{\mathbb{R}^N} &= (K_{0j}(u),\tilde{\psi})_{\mathbb{R}^N} = (\nabla K_{0j}(u),\nabla\Psi)_{\mathbb{R}^N} = (\mathrm{Div}(\mu_j^0 D(u)) - \nabla \mathrm{div} u,\nabla\Psi)_{\mathbb{R}^N} \\ &= -(\mu_j^0 D(u),\nabla^2\Psi)_{\mathbb{R}^N} + (\mathrm{div} u,\Delta\Psi)_{\mathbb{R}^N},\end{aligned}$$

and therefore by (6.28)

$$|(K_{0j}(u),\psi)_{R^N}| \le C\|\nabla u\|_{L_q(\mathbb{R}^N)} \|\psi\|_{L_{q'}(\mathbb{R}^N)},$$

which proves (6.24). This completes the proof of Lemma 6.7.

Lemma 6.8 Let $1 < q < \infty$. For $u \in H_q^2(\Omega_j)$ and $h \in H_q^3(\Omega_j)$, let $K_{1j}(u,h) \in H_q^1(\Omega_q) + \hat{H}_{q,0}^1(\Omega_j)$ be a unique solution of the weak Dirichlet problem (6.9). Then, we have

$$\begin{aligned}&\|K_{1j}(u,h)\|_{L_q(\Omega_j \cap B_j^1)} \\ &\le C(\|\nabla u\|_{L_q(\Omega_j)} + \|h\|_{H_q^2(\Omega_j)} + \|\nabla^2 u\|_{L_q(\Omega_j)}^{1/q} \|\nabla u\|_{L_q(\Omega_j)}^{1-1/q} + \|h\|_{H_q^3(\Omega_j)}^{1/q} \|h\|_{H^2(\Omega_j)}^{1-1/q}).\end{aligned}$$

Here, the constant C depends on q and C_K.

Remark 6.9 By Young's inequality, we have

$$\|K_{1j}(u,h)\|_{L_q(\Omega_j \cap B_j^1)} \le \varepsilon(\|\nabla^2 u\|_{L_q(\Omega_j)} + \|h\|_{H_q^3(\Omega_j)}) + C_\varepsilon(\|\nabla u\|_{L_q(\Omega_j)} + \|h\|_{H_q^2(\Omega_j)}) \tag{6.29}$$

for any $\varepsilon \in (0,1)$ with some constant $C_{\varepsilon,q}$ depending on ε and q.

Proof: For a proof, see Lemma 3.4 in Shibata[42]. To prove (6.23), we divide $V_2^1(\lambda)$ into two parts as $V_2^1(\lambda) = \nabla V_{21}^1(\lambda) + V_{22}^1(\lambda)$ where

$$V_{21}^1(\lambda)F = K(A_p(\lambda)F, B_p(\lambda)F) - \sum_{j=1}^\infty \zeta_j^0 K_{0j}(S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1) - \sum_{j=1}^\infty \zeta_j^1 K_{1j}(S_{1j}(\lambda)\tilde{\zeta}_j^0 F, H_j(\lambda)\tilde{\zeta}_j^1 F)),$$

$$V_{22}^1(\lambda)F = \sum_{j=1}^\infty (\nabla\zeta_j^0) K_{0j}(S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1) + \sum_{j=1}^\infty \nabla(\zeta_j^1) K_{1j}(S_{1j}(\lambda)\tilde{\zeta}_j^0 F, H_j(\lambda)\tilde{\zeta}_j^1 F)).$$

By (2.1), (2.2), (6.8), and (6.9), for any $\varphi \in \hat{H}_{q'0}^1(\Omega)$ we have $(\nabla V_{21}^1 F, \nabla\phi)_\Omega = I - II$, where

$$I = (\mathrm{Div}\,(\mu \mathrm{D}(A_p(\lambda)F) - \nabla \mathrm{div}(A_p(\lambda)F), \nabla\varphi)_\Omega,$$

$$\begin{aligned}II = &\sum_{j=1}^\infty ((\nabla\zeta_j^0) K_{0j}(S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1), \nabla(\varphi - e_j))_\Omega + \sum_{j=1}^\infty ((\nabla\zeta_j^1) K_{1j}(S_{1j}(\lambda)\tilde{\zeta}_j^1 F, H_j(\lambda)\tilde{\zeta}_j^1 F), \nabla\varphi)_\Omega \\ &+ \sum_{j=1}^\infty (\nabla K_{0j}(S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1), \nabla(\zeta_j^0(\varphi - e_j)))_\Omega + \sum_{j=1}^\infty (\nabla K_{1j}(S_{1j}(\lambda)\tilde{\zeta}_j^1 F, H_j(\lambda)\tilde{\zeta}_j^1 F), \nabla(\zeta_j^1\varphi))_\Omega \\ &- \sum_{j=1}^\infty ((\nabla\zeta_j^0)\nabla K_{0j}(S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1), \varphi - e_j)_\Omega - \sum_{j=1}^\infty ((\nabla\zeta_j^1)\nabla K_{1j}(S_{1j}(\lambda)\tilde{\zeta}_j^1 F, H_j(\lambda)\tilde{\zeta}_j^1 F), \varphi)_\Omega.\end{aligned}$$

Here and in the following, $e_j = c_j^0(\varphi)$ are constants given in Lemma 6.6. By the definition (6.17), we have

$$I = \sum_{j=1}^\infty (\mathrm{Div}(\mu \mathrm{D}(\zeta_j^0 S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1) - \nabla \mathrm{div}(\zeta_j^0 S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1), \nabla\varphi)_{\mathbb{R}^N}$$

$$+\sum_{j=1}^{\infty}(\mathrm{Div}(\mu\mathrm{D}(\zeta_j^1 S_{1j}(\lambda)\tilde{\zeta}_j^1 F)-\nabla \mathrm{div}(\zeta_j^1 S_{1j}(\lambda)\tilde{\zeta}_j^1 F),\nabla\varphi)_\Omega$$

$$=\sum_{j=1}^{\infty}(\zeta_j^0\mathrm{Div}(\mu_j^0\mathrm{D}(S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1))-\zeta_j^0\nabla\mathrm{div}(S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1),\nabla\varphi)_{\mathbb{R}^N}$$

$$+\sum_{j=1}^{\infty}(\zeta_j^1\mathrm{Div}(\mu_j^1 D(S_{1j}(\lambda)\tilde{\zeta}_j^1\mathrm{F}))-\zeta_j^1\nabla\mathrm{div}(S_{1j}(\lambda)\tilde{\zeta}_j^1\mathrm{F}),\nabla\phi)_\Omega+III,$$

where

$$III=\sum_{j=1}^{\infty}(\mathrm{Div}(\mu\mathrm{D}(\zeta_j^0 S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1))-\zeta_j^0\mathrm{Div}(\mu\mathrm{D}(S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1)),\nabla\varphi)_{\mathbb{R}^N}$$

$$-\sum_{j=1}^{\infty}(\nabla\mathrm{div}(\zeta_j^0 S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1)-\zeta_j^0\nabla\mathrm{div}(S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1),\nabla\varphi)_\Omega$$

$$+\sum_{j=1}^{\infty}(\mathrm{Div}(\mu\mathrm{D}(\zeta_j^1 S_{1j}(\lambda)\tilde{\zeta}_j^1 F))-\zeta_j^1\mathrm{Div}(\mu\mathrm{D}(S_{1j}(\lambda)\tilde{\zeta}_j^1 F)),\nabla\phi)_{\mathbb{R}^N}$$

$$-\sum_{j=1}^{\infty}(\nabla\mathrm{div}(\zeta_j^1 S_{1j}(\lambda)\tilde{\zeta}_j^1 F)-\zeta_j^1\nabla\mathrm{div}(S_{1j}(\lambda)\tilde{\zeta}_j^1 F),\nabla\varphi)_\Omega. \tag{6.30}$$

Since $\zeta_j^0(\varphi-e_j)\in\hat{H}^1_{q',0}(\mathbb{R}^N)$, and $\zeta_j^1\varphi\in\hat{H}^1_{q',0}(\Omega_j)$, by (6.8) and (6.9), we have

$$II=\sum_{j=1}^{\infty}(\zeta_j^0\mathrm{Div}(\mu_j^0 D(S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1))-\zeta_j^0\nabla\mathrm{div}(S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1),\nabla\phi)_{\mathbb{R}^N}$$

$$+\sum_{j=1}^{\infty}(\zeta_j^1\mathrm{Div}(\mu_j^1 D(S_{1j}(\lambda)\tilde{\zeta}_j^1 F))-\zeta_j^1\nabla\mathrm{div}(S_{1j}(\lambda)\tilde{\zeta}_j^1 F),\nabla\varphi)_\Omega+IV,$$

where we have set

$$IV=\sum_{j=1}^{\infty}\{2(K_{0j}(S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1)(\nabla\zeta_j^0),\nabla\phi)_\Omega+(K_{0j}(S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1)(\Delta\zeta_j^0),\phi-e_j)_\Omega$$
$$+2(K_{1j}(S_{1j}(\lambda)\tilde{\zeta}_j^1 F,H_j(\lambda)\tilde{\zeta}_j^1 F)(\nabla\zeta_j^1),\nabla\varphi)_\Omega+(K_{1j}(S_{1j}(\lambda)\tilde{\zeta}_j^1 F,S_j(\lambda)\tilde{\zeta}_j^1 F)(\Delta\zeta_j^1),\varphi)_\Omega$$
$$-(\mu_j^0 D(S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1):(\nabla^2\zeta_j^0),\varphi-e_j)_\Omega-(\mu_j^0 D(S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1)(\nabla\zeta_j^0),\nabla\varphi)_\Omega$$
$$+((\mathrm{div}S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1)(\Delta\zeta_j^0),\varphi-e_j)_\Omega+(\mathrm{div}S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1)(\nabla\zeta_j^0),\nabla\varphi)_\Omega$$
$$-(\mu_j^1 D(S_{1j}(\lambda)\tilde{\zeta}_j^1 F_1):(\nabla^2\zeta_j^1),\varphi)_\Omega-(\mu_j^1 D(S_{1j}(\lambda)\tilde{\zeta}_j^1 F_1)(\nabla\zeta_j^1),\nabla\varphi)_\Omega$$
$$+((\mathrm{div}S_{1j}(\lambda)\tilde{\zeta}_j^1 F_1)(\Delta\zeta_j^1),\varphi)_\Omega+(\mathrm{div}S_{1j}(\lambda)\tilde{\zeta}_j^1 F_1)(\nabla\zeta_j^1),\nabla\varphi)_\Omega.$$

Thus, we have

$$(\nabla V_{21}^1(\lambda)F,\nabla\phi)_\Omega=III+IV. \tag{6.31}$$

We let define operators $L(\lambda)$ and $M(\lambda)$ acting on $F\in Y_q(\Omega)$ by the following formulas:

$$L(\lambda)F$$
$$=\sum_{j=1}^{\infty}(\mathrm{Div}(\mu\mathrm{D}(\zeta_j^0 S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1))-\zeta_j^0\mathrm{Div}(\mu\mathrm{D}(S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1))$$
$$-\sum_{j=1}^{\infty}(\nabla\mathrm{div}(\zeta_j^0 S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1)-\zeta_j^0\nabla\mathrm{div}(S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1)$$
$$+\sum_{j=1}^{\infty}(\mathrm{Div}(\mu\mathrm{D}(\zeta_j^1 S_{1j}(\lambda)\tilde{\zeta}_j^1 F))-\zeta_j^1\mathrm{Div}(\mu\mathrm{D}(S_{1j}(\lambda)\tilde{\zeta}_j^1 F))$$
$$-\sum_{j=1}^{\infty}(\nabla\mathrm{div}(\zeta_j^1 S_{1j}(\lambda)\tilde{\zeta}_j^1 F)-\zeta_j^1\nabla\mathrm{div}(S_{1j}(\lambda)\tilde{\zeta}_j^1 F)+2\sum_{j=1}^{\infty}((\nabla\zeta_j^0)K_{0j}(S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1)$$
$$+2\sum_{j=1}^{\infty}((\nabla\zeta_j^1)K_{1j}(S_{1j}(\lambda)\tilde{\zeta}_j^1 F,H_j(\lambda)\tilde{\zeta}_j^1 F)-\sum_{j=1}^{\infty}(\mu_j^0\mathrm{D}(S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1)(\nabla\zeta_j^0)$$
$$+\sum_{j=1}^{\infty}(\mathrm{div}(S_{0j}(\lambda)\tilde{\zeta}_j^0 F_1)(\nabla\zeta_j^0)-\sum_{j=1}^{\infty}(\mu_j^1 D(S_{1j}(\lambda)\tilde{\zeta}_j^1 F)(\nabla\zeta_j^1)+\sum_{j=1}^{\infty}(\mathrm{div}(S_{1j}(\lambda)\tilde{\zeta}_j^1 F_1)(\nabla\zeta_j^1);$$

$$
\begin{aligned}
&<< \mathcal{M}(\lambda)F,\phi>> \\
&= -\sum_{j=1}^{\infty}(\mu_j^0 D(\mathcal{S}_{0j}(\lambda)\tilde{\zeta}_j^0 F_1):(\nabla^2\zeta_j^0),\varphi-e_j)_\Omega + \sum_{j=1}^{\infty}(\mathrm{div}(\mathcal{S}_{0j}(\lambda)\tilde{\zeta}_j^0 F_1)(\Delta\zeta_j^0),\varphi-e_j)_\Omega \\
&+\sum_{j=1}^{\infty}(K_{0j}(\mathcal{S}_{0j}(\lambda)\tilde{\zeta}_j^0 F_1)(\Delta\zeta_j^0),\varphi-e_j)_\Omega - \sum_{j=1}^{\infty}(\mu_j^1 \mathrm{D}(\mathcal{S}_{1j}(\lambda)\tilde{\zeta}_j^1 F):(\nabla^2\zeta_j^1),\varphi)_\Omega \\
&+\sum_{j=1}^{\infty}(\mathrm{div}(\mathcal{S}_{1j}(\lambda)\tilde{\zeta}_j^1 F)(\Delta\zeta_j^1),\varphi)_\Omega + \sum_{j=1}^{\infty}(K_{1j}(\mathcal{S}_{1j}(\lambda)\tilde{\zeta}_j^1 F,\mathcal{H}_j(\lambda)\tilde{\zeta}_j^1 F)(\Delta\zeta_j^1),\varphi)_\Omega .
\end{aligned}
$$

Here and in the following, $\hat{W}_{q,0}^{-1}(\Omega)$ denotes the dual space of $\hat{H}_{q',0}^{1}(\Omega)$, and $<<\cdot,\cdot>>$ denotes the duality between $\hat{W}_{q,0}^{-1}(\Omega)$ and $\hat{H}_{q',0}^{1}(\Omega)$.

Moreover, by (2.2) and (6.9), for $x\in\Gamma$ we have

$$
\begin{aligned}
&\mathcal{V}_{21}^1(\lambda)F =< \mu\mathrm{D}(\mathcal{A}_p(\lambda)F)\mathrm{n},\mathrm{n} > -(\mathbf{S}+\delta\Delta_\Gamma)\mathcal{B}_p(\lambda)F - \mathrm{div}\mathcal{A}_p(\lambda)F \\
&-\sum_{j=1}^{\infty}\zeta_j^1\{<(\mu(x_j^1)\mathrm{D}(\mathcal{S}_{1j}(\lambda)\tilde{\zeta}_j^1 F)\mathrm{n}_j,\mathrm{n}_j > -\delta(x_j^1)\Delta_{\Gamma_j}\mathcal{H}_j(\lambda)\tilde{\zeta}_j^1 F - \mathrm{div}\mathcal{S}_{1j}(\lambda)\tilde{\zeta}_j^1 F\} \\
&= \sum_{j=1}^{\infty}< \mu\mathrm{D}(\zeta_j^1\mathcal{S}_{1j}(\lambda)\tilde{\zeta}_j^1 F)\mathrm{n},\mathrm{n} > - \sum_{j=1}^{\infty}(\mathcal{B}+\delta\Delta_\Gamma)\zeta_j^1\mathcal{H}_j(\lambda)\tilde{\zeta}_j^1 F - \sum_{j=1}^{\infty}\mathrm{div}(\zeta_j^1\mathcal{S}_{1j}(\lambda)\tilde{\zeta}_j^1 F) \\
&-\sum_{j=1}^{\infty}\zeta_j^1\{<(\mu(x_j^1)\mathrm{D}(\mathcal{S}_{1j}(\lambda)\tilde{\zeta}_j^1 F)n_j,n_j > -\delta(x_j^1)\Delta_{\Gamma_j}\mathcal{H}_j(\lambda)\tilde{\zeta}_j^1 F - \mathrm{div}\mathcal{S}_{1j}(\lambda)\tilde{\zeta}_j^1 F\}.
\end{aligned}
$$

Thus, we define an operator $\mathcal{L}_b(\lambda)$ acting on $F\in Y_q(\Omega)$ by letting

$$
\begin{aligned}
\mathcal{L}_b(\lambda)F = \sum_{j=1}^{\infty}[&< \mu(x)(\mathrm{D}(\zeta_j^1\mathcal{S}_{1j}(\lambda)\tilde{\zeta}_j^1 F) - \zeta_j^1\mathrm{D}(\mathcal{S}_{1j}(\lambda)\tilde{\zeta}_j^1 F))n,n > - \mathcal{B}(\zeta_j^1\mathcal{H}_j(\lambda)\tilde{\zeta}_j^1 F) \\
&-\delta(\Delta_\Gamma(\zeta_j^1\mathcal{H}_j(\lambda)\tilde{\zeta}_j^1 F) - \zeta_j^1\Delta_\Gamma\mathcal{H}_j(\lambda)\tilde{\zeta}_j^1 F) - (\nabla\zeta_j^1)\mathcal{S}_{1j}(\lambda)\tilde{\zeta}_j^1 F],
\end{aligned}
$$

and then, $\mathcal{V}_{21}^1(\lambda)F = \mathcal{L}_b(\lambda)F$ on Γ.

We now prove the $\mathcal{R}$ boundedness of operator families $\mathcal{L}(\lambda)$, $\mathcal{M}(\lambda)$ and $\mathcal{L}_b(\lambda)$. We first prove that

$$
\mathcal{R}_{\mathcal{L}(Y_q(\Omega),\hat{W}_{q,0}^{-1}(\Omega))}(\{(\tau\partial_\tau)^\ell\mathcal{M}(\lambda)\mid\lambda\in\Lambda_{\sigma,\tilde{\lambda}_0}\}) \le (\varepsilon + C_{q,\varepsilon}\tilde{\lambda}_0^{-1/2})r_b \tag{6.32}
$$

for $\ell=0,1$. In fact, if we set

$$
\begin{aligned}
&<< \mathcal{M}_j^0(\lambda)F,\varphi>> = -(\mu(x_j^0)\mathrm{D}(\mathcal{S}_{0j}(\lambda)\tilde{\zeta}_j^0 F_1):(\nabla^2\zeta_j^0),\varphi-e_j)_\Omega + (\mathrm{div}(\mathcal{S}_{0j}(\lambda)\tilde{\zeta}_j^0 F_1)(\Delta\zeta_j^0),\varphi-e_j)_\Omega \\
&\qquad +(K_{0j}(\mathcal{S}_{0j}(\lambda)\tilde{\zeta}_j^0 F_1)(\Delta\zeta_j^0),\varphi-e_j)_\Omega, \\
&<< \mathcal{M}_j^1(\lambda)F,\varphi>> = -(\mu(x_j^1)D(\mathcal{S}_{1j}(\lambda)\tilde{\zeta}_j^1 F):(\nabla^2\zeta_j^1),\varphi)_\Omega \\
&\qquad +(\mathrm{div}(\mathcal{S}_{1j}(\lambda)\tilde{\zeta}_j^1 F)(\Delta\zeta_j^1),\varphi)_\Omega + (K_{1j}(\mathcal{S}_{1j}(\lambda)\tilde{\zeta}_j^1 F,\mathcal{H}_j(\lambda)\tilde{\zeta}_j^1 F)(\Delta\zeta_j^1),\varphi)_\Omega,
\end{aligned}
$$

then, by Lemma 6.6, Lemma 6.7 and (6.29), we have

$$
\begin{aligned}
&|<< \mathcal{M}_j^0(\lambda)F,\varphi>>| \le C_{M_2}\|\nabla\mathcal{S}_{0j}(\lambda)\tilde{\zeta}_j^0 F\|_{L_q(\mathbb{R}^N)}\|\nabla\varphi\|_{L_{q'}(B_j^0)}, \\
&|<< \mathcal{M}_j^1(\lambda)F,\varphi>>| \le \{\varepsilon(\|\mathcal{S}_{1j}\tilde{\zeta}_j^1 F\|_{H_q^2(\Omega_j)} + \|\mathcal{H}_j(\lambda)\tilde{\zeta}_j^1 F\|_{H_q^3(\Omega_j)}) \\
&\qquad +C_{\varepsilon,M_2}(\|\mathcal{S}_{1j}\tilde{\zeta}_j^1 F\|_{H_q^1(\Omega_j)} + \|\mathcal{H}_j(\lambda)\tilde{\zeta}_j^1 F\|_{H_q^2(\Omega_j)})\}\|\nabla\varphi\|_{L_q(B_j^1\cap\Omega)}.
\end{aligned}
$$

By (6.3), we have

$$
\sum_{j=1}^{\infty}\|\nabla\varphi\|_{L_{q'}(B_j^0)}^{q'} + \sum_{j=1}^{\infty}\|\nabla\varphi\|_{L_{q'}(B_j^1\cap\Omega)}^{q'} \le C_{q'}\|\nabla\varphi\|_{L_{q'}(\Omega)}^{q'}
$$

for any $\varphi\in\hat{H}_{q',0}^1(\Omega)$. By (6.13), (6.3), and Proposition (6.5), we have

$$
\begin{aligned}
&\sum_{j=1}^{\infty}\|\nabla\mathcal{S}_{0j}(\lambda)\tilde{\zeta}_j^0 F_1\|_{L_q(\mathbb{R}^N)}^q + \sum_{j=1}^{\infty}(\|\mathcal{S}_{1j}(\lambda)\tilde{\zeta}_j^1 F\|_{H_q^2(\Omega_j)}^q + \|\mathcal{H}_j(\lambda)\tilde{\zeta}_j^1 F\|_{H_q^3(\Omega_j)}^q) \\
&\le r_b^q(\sum_{j=1}^{\infty}\|\tilde{\zeta}_j^0 F_1\|_{L_q(\mathbb{R}^N)}^q + \sum_{j=1}^{\infty}\|\tilde{\zeta}_j^1 F\|_{CY_q(\Omega_j)}^q) \le r_b^q C_q\|F\|_{Y_q(\Omega)}^q < \infty.
\end{aligned}
$$

Thus, by Proposition 6.2, $\mathcal{M}(\lambda)F = \sum_{j=1}^{\infty}\mathcal{M}_j^0(\lambda)F + \sum_{j=1}^{\infty}\mathcal{M}_j^1(\lambda)F$ exists in $\hat{W}_{q,0}^{-1}(\Omega)$ for any $F\in Y_q(\Omega)$ and

$$\|\mathcal{M}(\lambda)F\|^q_{\hat{W}^{-1}_{q,0}(\Omega)} \le C^q_{M_2}\sum_{j=1}^\infty \|\nabla \mathcal{S}_{0j}(\lambda)\tilde{\zeta}^0_j F_1\|^q_{L_q(\mathbb{R}^N)} + \varepsilon^q\sum_{j=1}^\infty \|\mathcal{S}_{1j}(\lambda)\tilde{\zeta}^1_j F\|^q_{H^2_q(\Omega_j)}$$

$$+\varepsilon^q\sum_{j=1}^\infty \|\mathcal{H}_j(\lambda)\tilde{\zeta}^1_j F_j(\lambda)\tilde{\zeta}^1_j F\|^q_{H^3_q(\Omega_j)} + C^q_{\varepsilon,M_2}\sum_{j=1}^\infty \|\mathcal{S}_{1j}(\lambda)\tilde{\zeta}^1_j F\|^q_{H^1_q(\Omega_j)} + C^q_{\varepsilon,M_2}\sum_{j=1}^\infty \|\mathcal{H}_j(\lambda)\tilde{\zeta}^1_j F\|^q_{H^1_q(\Omega_j)}.$$

Analogously, by Proposition 6.2 we have

$$\left\|\sum_{\ell=1}^n r_\ell(u)\mathcal{M}(\lambda_\ell)F_\ell\right\|^q_{\hat{W}^{-1}_{q,0}(\Omega)} \le C^q_{M_2}\sum_{j=1}^\infty\left\|\sum_{\ell=1}^n r_\ell(u)\nabla\mathcal{S}_{0j}(\lambda_\ell)\tilde{\zeta}^0_j F_{1\ell}\right\|^q_{L_q(\mathbb{R}^N)}$$

$$+\varepsilon^q\sum_{j=1}^\infty\left\|\sum_{\ell=1}^n r_\ell(u)\mathcal{S}_{1j}(\lambda_\ell)\tilde{\zeta}^1_j F_\ell\right\|^q_{H^2_q(\Omega_j)} + \varepsilon^q\sum_{j=1}^\infty\left\|\sum_{\ell=1}^n r_\ell(u)\mathcal{H}_j(\lambda_\ell)\tilde{\zeta}^1_j F_\ell\right\|^q_{H^3_q(\Omega_j)}$$

$$+C^q_{\varepsilon,M_2}\sum_{j=1}^\infty\left\|\sum_{\ell=1}^n r_\ell(u)\mathcal{S}_{1j}(\lambda_\ell)\tilde{\zeta}^1_j F_\ell\right\|^q_{H^1_q(\Omega_j)} + C^q_{\varepsilon,M_2}\sum_{j=1}^\infty\left\|\sum_{\ell=1}^n r_\ell(u)\mathcal{H}_j(\lambda_\ell)\tilde{\zeta}^1_j F_\ell\right\|^q_{H^1_q(\Omega_j)}.$$

Noting that $\Omega\cap B^1_j = \Omega_j\cap B^1_j$, by (6.13), (6.5), Proposition 6.5, and Proposition (3.4), we have

$$\int_0^1\left\|\sum_{\ell=1}^n r_\ell(u)\mathcal{M}(\lambda_\ell)F\right\|^q_{\ell\hat{W}^{-1}_{q,0}(\Omega)} du \le C^q_{M_2}\tilde{\lambda}_0^{-q/2} r_b^q\int_0^1\sum_{j=1}^\infty\left\|\sum_{\ell=1}^n r_\ell(u)\tilde{\zeta}^0_j F_{1\ell}\right\|^q_{L_q(\mathbb{R}^N)} du$$

$$+\varepsilon^q r_b^q\int_0^1\sum_{j=1}^\infty\left\|\sum_{\ell=1}^n r_\ell(u)\tilde{\zeta}^1_j F_\ell\right\|^q_{\mathcal{Y}_q(\Omega_j)} du + C^q_{\varepsilon,M_2}\tilde{\lambda}_0^{-q/2} r_b^q\int_0^1\sum_{j=1}^\infty\left\|\sum_{\ell=1}^n r_\ell(u)\tilde{\zeta}^1_j F_\ell\right\|^q_{\mathcal{Y}_q(\Omega_j)} du$$

$$\le C_q(\varepsilon^q + C^q_{\varepsilon,M_2}\tilde{\lambda}_0^{-q/2})r_b^q\int_0^1\left\|\sum_{\ell=1}^n r_\ell(u)F_\ell\right\|^q_{\mathcal{Y}_q(\Omega)} du,$$

which shows (6.32). Analogously, we can prove

$$\begin{aligned}&\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega),L_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell\mathcal{L}(\lambda)\mid\lambda\in\Lambda_{\sigma,\tilde{\lambda}_0}\})\le C_{M_2}r_b\tilde{\lambda}_0^{-1/2},\\&\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega),H^1_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell\mathcal{L}_b(\lambda)\mid\lambda\in\Lambda_{\sigma,\tilde{\lambda}_0}\})\le C_{M_2}r_b\tilde{\lambda}_0^{-1/2}\end{aligned} \tag{6.33}$$

for $\ell=0,1$.

We now use the following lemma.

Lemma 6.10: Let $1<q<\infty$. Then, there exists a linear map E from $\hat{W}^{-1}_{q,0}(\Omega)$ into $L_q(\Omega)^N$ such that for any $F\in\hat{W}^{-1}_{q,0}(\Omega)$, $\|E(F)\|_{L_q(\Omega)}\le C\|F\|_{\hat{W}^{-1}_{q,0}(\Omega)}$ and

$$<F,\varphi>=(E(F),\nabla\varphi)_\Omega \quad \text{for all } \varphi\in\hat{\mathrm{H}}^1_{q',0}(\Omega).$$

Proof: The lemma follows from the Hahn-Banach theorem by indentifying $\hat{H}^1_{q',0}(\Omega)$ with a closed subspace of $L_{q'}(\Omega)^N$ via the mapping: $\varphi\mapsto\nabla\varphi$. Applying Lemma 6.10 and using (6.31) and (6.32), we have

$$(\nabla V^1_{21}(\lambda)F,\nabla\varphi)_\Omega = (\mathcal{L}(\lambda)F + E(\mathcal{M}(\lambda)F),\nabla\varphi)_\Omega \quad \text{for all } \varphi\in\hat{\mathrm{H}}^1_{q',0}(\Omega), \tag{6.34}$$

subject to $V^1_{21}(\lambda)F=\mathcal{L}_b(\lambda)F$ on Γ, and

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega),L_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell E\circ\mathcal{M}(\lambda)\mid\lambda\in\Sigma_{\sigma,\tilde{\lambda}_0}\})\le C(\varepsilon + C_{q,\varepsilon}\tilde{\lambda}_0^{-1/2})r_b, \tag{6.35}$$

where $E\circ\mathcal{M}(\lambda)$ denotes a bounded linear operator family acting on F by $E\circ\mathcal{M}(\lambda)F = E(\mathcal{M}(\lambda)F)$. By Remark 1.5, we have $V^1_{21}(\lambda)F = \mathcal{L}_b(\lambda)F + \mathcal{K}_0(\mathcal{L}(\lambda)F + E(\mathcal{M}(\lambda)F) - \nabla\mathcal{L}_b(\lambda)F)$, and so by (6.33) and (6.35), we see that $\nabla V^1_{21}(\lambda)\in \mathrm{Hol}(\Sigma_{\sigma,\tilde{\lambda}_0},\mathcal{L}(\mathcal{Y}_q(\Omega),L_q(\Omega)^N))$ and

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega),L_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell\nabla V^1_{21}(\lambda)\mid\lambda\in\Sigma_{\sigma,\tilde{\lambda}_0}\})\le C_q(\varepsilon + C_{M_2,\varepsilon}\tilde{\lambda}_0^{-1/2})r_b \tag{6.36}$$

for $\ell=0,1$.

Finally, by Lemma 6.7, (6.29), (6.5), and Proposition 6.2, we have

$$V^1_{22}(\lambda)\in\mathrm{Hol}(\Sigma_{\sigma,\tilde{\lambda}_0},\mathcal{L}(\mathcal{Y}_q(\Omega),L_q(\Omega)^N)),$$

$$R_{L(Y_q(\Omega),L_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell V_{21}^1(\lambda)\mid\lambda\in\Sigma_{\sigma,\tilde\lambda_0}\})\le C_q(\varepsilon+C_\varepsilon\tilde\lambda_0^{-1/2})r_b$$

for $\ell=0,1$, which, combined with (6.36) and the formula: $V_2^1(\lambda)=\nabla V_{21}^1(\lambda)+V_{22}^2(\lambda)$, leads to (6.23).

Proof of theorem 2.1, existence part

Choosing ε so small that $C_{q,r}r_b\varepsilon\le 1/4$, and $\tilde\lambda_0$ so large that

$$C_q r_b C_{M_2,\varepsilon}(\tilde\lambda_0^{-1}\gamma_\sigma+\tilde\lambda_0^{-1/2})\le 1/4 \tag{6.37}$$

in (6.20), we have

$$R_{L(Y_q(\Omega))}(\{(\tau\partial_\tau)^\ell \mathrm{F}_\lambda V(\lambda)\mid\lambda\in\Lambda_{\sigma,\tilde\lambda_0}\})\le 1/2 \tag{6.38}$$

for $\ell=0,1$. Let λ_* be a large number for which $\lambda_*\ge(8C_q r_b C_{M_2,\varepsilon})^2$, and then setting $\tilde\lambda_0=\lambda_*\gamma_\sigma$, we have (6.37). By (6.37), $(I-\mathrm{F}_\lambda V(\lambda))^{-1}=\sum_{j=1}^{\infty}(\mathrm{F}_\lambda V(\lambda))^j$ exists in $Hol(\Sigma_{\sigma,\tilde\lambda_0},L(Y_q(\Omega)))$ and

$$R_{L(Y_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\mathrm{I}-\mathrm{F}_\lambda V(\lambda))^{-1}\mid\lambda\in\Lambda_{\sigma,\tilde\lambda_0}\})\le 4 \tag{6.39}$$

for $\ell=0,1$. Moreover, by (6.19) and (6.38)

$$\|\mathrm{F}_\lambda V(\lambda)(f,d,\mathrm{h})\|_{Y_q(\Omega)}\le(1/2)\|F_\lambda(f,d,h)\|_{Y_q(\Omega)}. \tag{6.40}$$

Since $\|F_\lambda(f,d,h)\|_{Y_q(\Omega)}$ gives an equivalent norm in $Y_q(\Omega)$ for $\lambda\ne 0$, by (6.40) $(I-V(\lambda))^{-1}=\sum_{j=0}^{\infty}V(\lambda)^j$ exists in $L(Y_q(\Omega))$. Since $u=A_p(\lambda)\mathrm{F}_\lambda(f,d,h)$ and $h=B_p(\lambda)F_\lambda(f,d,h)$ satisfy Eq. (6.16), setting

$$v=A_p(\lambda)\mathrm{F}_\lambda(\mathrm{I}-V(\lambda))^{-1}(f,d,h),\quad \rho=A_p(\lambda)(\lambda)\mathrm{F}_\lambda(\mathrm{I}-V(\lambda))^{-1}(f,d,h),$$

we see that $v\in H_q^2(\Omega)^N$, $\rho\in H_q^3(\Omega)$ and v and ρ satisfy the equations:

$$\begin{cases}\lambda v-\mathrm{Div}(\mu\mathrm{D}(v)-K(v,\rho)\mathrm{I}) & =f & \text{in }\Omega,\\ \lambda\rho+A_\sigma\cdot\nabla_\Gamma\rho-v\cdot n+F\ v & =d & \text{on }\Gamma,\\ (\mu\mathrm{D}(v)-K(v,\rho)\mathrm{I}-((B+\delta\Delta_\Gamma)\rho)\mathrm{I})n & =h & \text{on }\Gamma,\end{cases} \tag{6.41}$$

Moreover, by (6.19) we have $\mathrm{F}_\lambda(\mathrm{I}-V(\lambda))^{-1}=(I-F_\lambda V(\lambda))^{-1}F_\lambda$. Thus, setting

$$A_r(\lambda)=A_p(\lambda)(I-F_\lambda V(\lambda))^{-1},\quad H_r(\lambda)=B_p(\lambda)(I-F_\lambda V(\lambda))^{-1}$$

we see that $v=A_r(\lambda)F_\lambda(f,d,h)$ and $\rho=H_r(\lambda)F_\lambda(f,d,h)$ are solutions of Eq.(1.6). Since we may assume that $\lambda_*\gamma_\sigma\ge\lambda_1$ in (6.18), by (6.18) and (6.39), we have

$$R_{L(Y_q(\Omega),H_q^{2-j}(\Omega)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}A_r(\lambda))\mid\lambda\in\Lambda_{\sigma,\lambda_*\gamma_\sigma}\})\le C_q r_b,$$

$$R_{L(Y_q(\Omega),H_q^{3-k}(\Omega))}(\{(\tau\partial_\tau)^\ell(\lambda^k H_r(\lambda))\mid\lambda\in\Lambda_{\sigma,\lambda_*\gamma_\sigma}\})\le C_q r_b,$$

for $\ell=0,1$, $j=0,1,2$ and $k=0,1$. This completes the proof of the existence part of Theorem 2.1.

Uniqueness, a proof of theorem 1.10

In this subsection, we shall prove Theorem 1.10. Let $u\in H_q^2(\Omega)^N$, $q\in H_q^1(\Omega)+\hat H_{q,0}^1(\Omega)$, and $h\in H_q^3(\Omega)$ satisfy the homogeneous equations:

$$\begin{cases}\lambda u-\mathrm{Div}(\mu\mathrm{D}(u)-q\mathrm{I}),\ \ \mathrm{div}\,u & =0 & \text{in }\Omega,\\ \lambda h-u\cdot n & =0 & \text{on }\Gamma,\\ (\mu\mathrm{D}(u)-qI-\delta(\Delta_\Gamma)h)\mathrm{I})n & =0 & \text{on }\Gamma,\end{cases} \tag{6.42}$$

where δ is a positive constant. We shall prove that $u=0$ and $h=0$ below. Let λ_0 be a large positive number such that for any $\lambda\in\Sigma_{\varepsilon,\lambda_0}$ the existence theorem holds with $q'=(q-1)/q$. Let $J_{q'}(\Omega)$ be a solenoidal spaces defined in (1.7) and let g be any element in $J_{q'}(\Omega)$. Let $v\in H_{q'}^2(\Omega)^N$, $p\in H_{q'}^1(\Omega)+\hat H_{q',0}^1(\Omega)$, and $\rho\in H_{q'}^3(\Gamma)$ be solutions to the equations:

$$\begin{array}{lll}\bar\lambda v-\mathrm{Div}(\mu\mathrm{D}(v)-p)\mathrm{I}) & =g & \text{in }\Omega,\\ \bar\lambda\rho-n\cdot v & =0 & \text{on }\Gamma,\\ (\mu\mathrm{D}(v)-p)\mathrm{I})n-((\tau+\delta\Delta_\Gamma)\rho)n & =0 & \text{on }\Gamma.\end{array} \tag{6.43}$$

Let $K(v,\rho)\in H_{q'}^1(\Omega)+\hat H_{q',0}^1(\Omega)$ be a solution of the weak Dirichlet problem:

$$(\nabla K(v,\rho),\nabla\varphi)_\Omega = (\mathrm{D}iv(\mu \mathrm{D}(v)) - \nabla \mathrm{d}iv\mathrm{v},\nabla\varphi)_\Omega \quad \text{for any} \varphi\in \hat{\mathrm{H}}^1_{\mathrm{q},0}(\Omega), \tag{6.44}$$

subject to $K(v,\rho) =< \mu\mathrm{D}(v)n,n> -\delta\Delta_\Gamma\rho - \mathrm{d}iv v$ on Γ. And then, as was seen in Subsect. 2.1, $p = K(v,\rho)$. This facts yields that $v\in J_{q'}(\Omega)$. In fact, for any $\varphi\in\hat{H}^1_{q,0}(\Omega)$, we have

$$\begin{aligned} 0 &= (\mathrm{g},\nabla\varphi)_\Omega = \bar{\lambda}(\mathrm{v},\nabla\varphi)_\Omega - (\mathrm{D}iv(\mu\mathrm{D}(\mathrm{v})),\nabla\varphi)_\Omega + (\nabla K(\mathrm{v},\rho),\nabla\varphi)_\Omega \\ &= \bar{\lambda}(\mathrm{v},\nabla\varphi)_\Omega - (\nabla\mathrm{d}iv\mathrm{v},\nabla\varphi)_\Omega. \end{aligned} \tag{6.45}$$

Since $H^1_{q,0}(\Omega)\subset\hat{H}^1_{q,0}(\Omega)$, for any $\varphi\in H^1_{q,0}(\Omega)$, we have

$$0 = \bar{\lambda}(\mathrm{d}iv\mathrm{v},\varphi)_\Omega + (\nabla\mathrm{d}iv\mathrm{v},\nabla\varphi)_\Omega.$$

Choose $\lambda_0>0$ larger if necessary, we may assume that the uniquness of the resolvent problem for the weak Laplace-Dirichlet operator holds, and so $\mathrm{d}iv\mathrm{v}=0$. Putting this and (6.45) together gives $(v,\nabla\varphi)_\Omega = 0$ for any $\varphi\in\hat{H}^1_{q,0}(\Omega)$, that is $\mathrm{v}\in J_{q'}(\Omega)$. Moreover, by Definition 1.6, $u\in J_q(\Omega)$.

Since $p\in H^1_{q'}(\Omega)+\hat{H}^1_{q',0}(\Omega)$, we write $p = A_1 + A_2 \in H^1_{q'}(\Omega)+\hat{H}^1_{q',0}(\Omega)$. And then, by the divergence theorem of Gauss

$$(u,\nabla p)_\Omega = (u\cdot n, p)_\Gamma - (\mathrm{d}iv u, A_1)_\Omega = (u\cdot n,p)_\Gamma$$

because $u\in J_q(\Omega)$, $A_1\in H^1_q(\Omega)$, and $A_2 = 0$ on Γ. Thus, by the divergence theorem of Gauss we have

$$\begin{aligned} (\mathrm{u},\mathrm{g})_\Omega &= \lambda(\mathrm{u},v) - (u,\mathrm{D}iv(\mu\mathrm{D}(v)-p\mathrm{I}))_\Omega \\ &= \lambda(\mathrm{u},\mathrm{v}) - (\mathrm{u},(\mu\mathrm{D}(v)-p)n)_\Gamma + (\frac{\mu}{2}\mathrm{D}(\mathrm{u}),\mathrm{D}(\mathrm{v}))_\Omega \\ &= \lambda(\mathrm{u},\mathrm{v}) - (\mathrm{u}\cdot\mathrm{n},\delta\Delta_\Gamma\rho)_\Gamma + (\frac{\mu}{2}D(u),\mathrm{D}(\mathrm{v}))_\Omega. \end{aligned}$$

Since $\lambda h = \mathrm{u}\cdot\mathrm{n}$, we have

$$(\mathrm{u},\mathrm{g})_\Omega = \lambda(\mathrm{u},\mathrm{v})_\Omega + \lambda\delta(\nabla_\Gamma h,\nabla_\Gamma\rho)_\Gamma + (\frac{\mu}{2}\mathrm{D}(\mathrm{u}),\mathrm{D}(\mathrm{v}))_\Omega. \tag{6.46}$$

Analogously, we have

$$0 = (\lambda u - \mathrm{D}iv(\mu\mathrm{D}(\mathrm{u}) - q\mathrm{I}), v)_\Omega = \lambda(\mathrm{u},\mathrm{v})_\Omega + \lambda\delta(\nabla_\Gamma h,\nabla_\Gamma\rho)_\Gamma + (\frac{\mu}{2}\mathrm{D}(\mathrm{u}),\mathrm{D}(\mathrm{v}))_\Omega,$$

which, combined with (6.46), leads to

$$(u,g)_\Omega = 0 \quad \text{for any } \mathrm{g}\in \mathrm{J}_{\mathrm{q}'}(\Omega). \tag{6.47}$$

For any $f\in C^\infty_0(\Omega)^N$, let $\psi\in\hat{H}^1_{q',0}(\Omega)$ be a solution to the variational equation $(f,\nabla\varphi)_\Omega = (\nabla\psi,\nabla\varphi)_\Omega$ for any $\varphi\in\hat{H}^1_{q,0}(\Omega)$. Let $g = f - \nabla\psi$, and then $g\in J_{q'}(\Omega)$ and $(\mathrm{u},\nabla\psi)_\Omega = 0$. Thus, by (6.47), $(\mathrm{u},f)_\Omega = (\mathrm{u},g)_\Omega = 0$, which, combined with the arbitrariness of the choice of f, leads to $\mathrm{u}=0$. And then, by the second equation of (6.42) yields that $h=0$. This completes the proof of Theorem 1.10.

A priori estimate

In this section, we consider the uniqueness. If $A_\sigma = 0$, F $\mathrm{v}=0$, and B $\rho = 0$, then, as was seen in Subsec:6.5, we can show the uniqueness of solutions by using the existence of solutions of the dual problem. But, in the general case, we can not find a suitable dual problem, and so to prove the uniqueness we derive a priori estimates. For this purpose, we have to restrict our domain Ω slightly. We introduce the notion of finite covering domains.

Definition 7.1: Let $k=2$ or 3 and let Ω be a domain in $\mathbb{R}^N$. We say that Ω is a uniformly C^k domain whose inside is finitely covering if Ω is a uniformly C^k domain and the following condition holds: v Let $\{\zeta^i_j\}^\infty_{j=1}$ $(i=0,1)$ be the partition of unity given in Proposition 6.1. Let

$$O = \{(\bigcup_{j=1}^\infty supp\nabla\zeta^0_j)\cup(\bigcup_{j=1}^\infty supp\nabla\zeta^1_j)\}\cap\Omega.$$

Then, there exist a finite number of subdomains O_j $(j=1,\ldots,\iota)$ such that $O\subset\bigcup^\iota_{j=1}O_j$ and each O_j satisfies one of the following conditions:

a) There exists an $R>0$ such that $O_j\subset\Omega_R$, where $\Omega_R = \{x\in\Omega \mid |x|<R\}$,

b) There exist a translation τ, a rotation A, a domain $D\subset\mathbb{R}^{N-1}$, a coordinate functions $a(x')$ defined for $x'\in\mathrm{D}$, and a positive constant b such that $0\le a(x')<b$ for $x\in D$, $A\circ\tau(O_j)\subset\{x=(x',x_N) \mid x'\in\mathrm{D}, a(x')\le x_N\le b\}\subset A\circ\tau(\Omega)$, $\{x=(x',x_N)\in\mathbb{R}^N \mid x'\in D, x_N = a(x')\}\subset A\circ\tau(\Gamma)$.

Where, for any subset E of $\mathbb{R}^N$, $A(E)=\{Ax \mid x\in E\}$ with some orthogonal matrix A and $\tau(E)=\{x+y \mid x\in E\}$ with some $y\in\mathbb{R}^N$.

Example 7.2: Let Ω be a domain whose boundary Γ is a C^k hypersurface. If Ω satisfies one of the following conditions, then Ω is a uniformly C^k domain whose inside is finite covering.

(1) Ω is bounded, or Ω is an exterior domain, that is, $\Omega=\mathbb{R}^N\setminus\overline{O}$ with some bounded domain O.

(2) $\Omega=\mathbb{R}^N_+$ (half space), or Ω is a perturbed half space, that is, there exists an $R>0$ such that $\Omega\cap B^R=\mathbb{R}^N_+\cap B^R$, where $B^R=\{x\in\mathbb{R}^N \mid |x|>R\}$.

(3) Ω is a layer L or perturbed layer, that is, there exists an $R>0$ such that $\Omega\cap B^R=L\cap B^R$. Here $L=\{x=(x',x_N)\in\mathbb{R}^N \mid x'=(x_1,\ldots,x_{N-1})\in R^{N-1}, a<x_N<b\}$ for some constants a and b for which $a<b$.

(4) Ω is a tube, that is, there exists a bounded domain D in $\mathbb{R}^{N-1}$ such that $\Omega=D\times R$.

(5) There exist an $R>0$ and several orthogonal transforms, R_i ($i=1,\ldots,M$), such that $\Gamma\cap B^R=(\bigcup_{i=1}^M R_i R_0^N)\cap B^R$.

(6) There exist an $R>0$, half tubes, T_i $(i=1,\ldots,M)$, and orthogonal transforms, R_i $(i=1,\ldots,M)$, such that $\Omega\cap B^R=(\bigcup_{i=1}^M R_i T_i)\cap B^R$, where what T_i is a half tube means that $T_i=D_i\times[0,\infty)$ with some bounded domain D_i of R^{N-1}.

In this section, under the finite covering assumption, (6.2), and (??), we prove a priori estimates of Eq. (2.4), and as a result, we have the uniqueness of solutions. The following theorem is the main result of this section.

Theorem 7.3: Let $1<q<\infty$. Let Ω be a uniformly C^3 domain whose inside is finite covering. Then, there exists a $\lambda_0>0$ such that for any $\lambda\in\Lambda_{\sigma,\lambda_0\gamma_\sigma}$ and $(\mathrm{u},h)\in H_q^2(\Omega)^N\times H_q^3(\Omega)$ satisfying Eq. (2.4), we have

$$\begin{aligned}&|\lambda|\,\|u\|_{L_q(\Omega)}+|\lambda|^{1/2}\|u\|_{H_q^1(\Omega)}+\|u\|_{H_q^2(\Omega)}+|\lambda|\,\|h\|_{H_q^2(\Omega)}+\|h\|_{H_q^3(\Omega)}\\ &\le C\{\|f\|_{L_q(\Omega)}+\|d\|_{W_q^{2-1/q}(\Gamma)}+|\lambda|^{1/2}\|h\|_{L_q(\Omega)}+\|h\|_{H_q^1(\Omega)}\}.\end{aligned}\tag{7.1}$$

Corollary 7.4: Let $1<q<\infty$. Let Ω be a uniformly C^3 domain whose inside is finite covering. Then, there exists a $\lambda_0>0$ such that the uniqueness holds for Eq. (1.6) for any $\lambda\in\Lambda_{\sigma,\lambda_0\gamma_\sigma}$.

In what follows, we shall prove Theorem 7.3. We use the same notation as in Sect.6. Let $u_j^i=\zeta_j^i u$ and $h_j=\zeta_j^1 h$. And then, u_j^0 satisfy the equations:

$$\lambda \mathrm{u}_j^0-\mathrm{Div}(\mu(x_j^0)\mathrm{D}(u_j^0)-K_{0j}(\mathrm{u}_j^0)\mathrm{I})=f_j^0 \quad \text{in } \mathbb{R}^{\mathrm{N}}. \tag{7.2}$$

And also, u_j^1 and h_j satisfy the equations:

$$\begin{cases}\lambda u_j^1-\mathrm{Div}(\mu(x_j^1)\mathrm{D}(\mathrm{u}_j^1)-K_{1j}(u_j^1,h_j^1)\mathrm{I}) &=f_j^1 &\text{in } \Omega_{\mathrm{j}},\\ \lambda h_j+A_\sigma(x_j^1)\cdot\nabla_{\Gamma_j}h_j-\mathrm{n}_j\cdot\mathrm{u}_j &=d_j &\text{on } \Gamma_{\mathrm{j}},\\ (\mu(x_j^1)\mathrm{D}(\mathrm{u}_j^1)-K_{1j}(\mathrm{u}_j^1,h_j^1)\mathrm{I})n_j-\delta(x_j^1)(\Delta_{\Gamma_j}h_j)n_j &=h_j &\text{on } \Gamma_{\mathrm{j}}.\end{cases}\tag{7.3}$$

Where, we have set

$$\begin{aligned}f_j^0&=\zeta_j^0 f+\zeta_j^0\mathrm{Div}(\mu(x)\mathrm{D}(\mathrm{u}))-\mathrm{Div}(\mu(x)\mathrm{D}(\zeta_j^0\mathrm{u}))+\mathrm{Div}((\mu(x)-\mu(x_j^0))\mathrm{D}(\zeta_j^0\mathrm{u}))\\ &\quad-(\zeta_j^0\nabla K(\mathrm{u},h)-\nabla K_{0j}(\zeta_j^0\mathrm{u}));\\ f_j^1&=\zeta_j^1 f+\zeta_j^1\mathrm{Div}(\mu(x)\mathrm{D}(\mathrm{u}))-\mathrm{Div}(\mu(x)\mathrm{D}(\zeta_j^1 u))+\mathrm{Div}((\mu(x)-\mu(x_j^1))\mathrm{D}(\zeta_j^1 u))\\ &\quad-(\zeta_j^1\nabla K(\mathrm{u},h)-\nabla K_{1j}(\zeta_j^1\mathrm{u},\zeta_j^1 h));\\ d_j&=\zeta_j^1 d-\zeta_j^1(A_\sigma(x)-A_\sigma(x_j^1))\cdot\nabla_{\Gamma_j}h_j-A_\sigma(x_j^1)\cdot(\zeta_j^1\nabla_{\Gamma_j}h-\nabla_{\Gamma_j}(\zeta_j^1 h))-\zeta_j^1 F\,u;\\ h_j&=\zeta_j^1 h-\{\zeta_j^1(\mu(x)-\mu(x_j^1))\mathrm{D}(u)+\mu(x_j^1)(\zeta_j^1\mathrm{D}(u)-\mathrm{D}(\zeta_j^1 u))\}n+(\zeta_j^1 K(\mathrm{u},h)-K(\zeta_j^1\mathrm{u},\zeta_j^1 h))n\\ &\quad+\zeta_j^1(B\,h)\mathrm{n}+\zeta_j^1(\delta(x)-\delta(x_j^1))(\Delta_\Gamma h)\mathrm{n}+\delta(x_j^1)(\zeta_j^1\Delta_\Gamma h-\Delta_\Gamma(\zeta_j^1 h))\mathrm{n}.\end{aligned}$$

Set

$$E_\lambda(u,h)=|\lambda|^q\|u\|_{L_q(\Omega)}^q+|\lambda|^{q/2}\|u\|_{H_q^1(\Omega)}^q+\|u\|_{H_q^2(\Omega)}^q+|\lambda|^q\|h\|_{H_q^2(\Omega)}^q+\|h\|_{H_q^3(\Omega)}^q.$$

Employing the similar argument to that in Subsec. 6.3, for any positive number ω we have

$$\begin{aligned} E_\lambda(u,h) &\leq C\sum_{j=1}^{\infty}(\|f_j^0\|^q_{L_q(\mathbb{R}^N)}+\|f_j^1\|^q_{L_q(\Omega_j)}+\|d_j\|^q_{W_q^{2-1/q}(\Gamma_j)}+|\lambda|^{q/2}\|h_j\|^q_{L_q(\Omega_j)}+\|h_j\|^q_{H_q^1(\Omega_j)}\} \\ &\leq C\{\|f\|^q_{L_q(\Omega)}+\|d\|^q_{W_q^{2-1/q}(\Gamma)}+|\lambda|^{q/2}\|h\|_{L_q(\Omega)}+\|h\|^q_{H_q^1(\Omega)}+\gamma_\sigma^q\|h\|^q_{H_q^2(\Omega)} \\ &\quad+(\omega^q+M_1^q)(\|u\|^q_{H_q^2(\Omega)}+\|h\|^q_{H_q^3(\Omega)}+|\lambda|^{q/2}\|h\|_{H_q^1(\Omega)}) \\ &\quad+C_{\omega,M_2}(\|u\|_{H_q^1(\Omega)}+|\lambda|^{q/2}\|u\|_{L_q(\Omega)}+\|h\|^q_{H_q^2(\Omega)}+|\lambda|^{q/2}\|h\|^q_{H_q^1(\Omega)})+\|K(u,h)\|^q_{L_q(O)}\}. \end{aligned} \tag{7.4}$$

Where, C_{ω,M_2} is a constant depending on ω and M_2, and we have used the assumption that

$$(\bigcup_{j=0}^{\infty} supp\nabla\zeta_j^0)\cup(\bigcup_{j=1}^{\infty} supp\nabla\zeta_j^1)=O.$$

To estimate $\|K(u,h)\|_{L_q(O)}$, we need the following Poincarés' type lemma.

Lemma 7.5: Let $1<q<\infty$ and let Ω be a uniformly C^2 domain whose inside is finite covering. Let O be a set given in Definition 7.1. Then, we have

$$\|\varphi\|_{L_q(O)}\leq C_{q,O}\|\nabla\varphi\|_{L_q(\Omega)} \quad \text{for any } \varphi\in\hat{H}^1_{q,0}(\Omega)$$

with some constant $C_{q,O}$ depending solely on O and q.

Proof: A proof of this lemma is given in Appendix 11 below. We now prove that for any $\omega>0$ there exists a constant C_{ω,M_2} depending on ω and M_2 such that

$$\|K(u,h)\|_{L_q(O)}\leq\omega(\|u\|_{H_q^2(\Omega)}+\|h\|_{H_q^3(\Omega)})+C_{\omega,M_2}(\|u\|_{H_q^1(\Omega)}+\|h\|_{H_q^2(\Omega)}). \tag{7.5}$$

For this purpose, we estimate $|(K(u,h),\psi)_\Omega|$ for any $\psi\in C_0^\infty(O)$. By Lemma 7.5,

$$|(\varphi,\psi)_\Omega|\leq\|\varphi\|_{L_q(O)}\|\psi\|_{L_{q'}(O)}\leq C_{q,O}\|\nabla\varphi\|_{L_q(O)}\|\psi\|_{L_{q'}(O)}$$

for any $\varphi\in\hat{H}^1_{q,0}(\Omega)$. Thus, by the Hahn-Banach theorem, there exists a $g\in L_{q'}(\Omega)^N$ such that $\|g\|_{L_{q'}(\Omega)}\leq C_{q',O}\|\psi\|_{L_{q'}(O)}$ and

$$(\varphi,\psi)_\Omega=(\nabla\varphi,g)_\Omega \tag{7.6}$$

for any $\varphi\in\hat{H}^1_{q,0}(\Omega)$. In particular, $\mathrm{div}\, g=-\psi$, and therefore $\|\mathrm{div}\, g\|_{L_{q'}(\Omega)}\leq\|\psi\|_{L_{q'}(O)}$. By the assumption of the unique existence of solutions of the weak Dirichlet problem and its regularity theorem, Theorem 10.1 in Appendix 10 below, there exists a $\Psi\in\hat{H}^1_{q',0}(\Omega)$ such that $\nabla^2\Psi\in L_q(\Omega)^N$, Ψ satisfies the weak Dirichlet problem:

$$(\nabla\Psi,\nabla\varphi)_\Omega=(g,\nabla\varphi)_\Omega \quad \text{for any} \varphi\in\hat{H}^1_{q,0}(\Omega) \tag{7.7}$$

and the estimate:

$$\|\nabla\Psi\|_{H^1_{q'}(\Omega)}\leq C_{q,O}\|\psi\|_{L_{q'}(O)}. \tag{7.8}$$

Let $L=K(u,h)-\{<\mu D(u)n,n>-(B+\sigma\Delta_\Gamma)h-\mathrm{div}u\}$, and then $L\in\hat{H}^1_{q,0}(\Omega)$. Thus, by (7.6), (7.7) with $\varphi=L$ and the divergence theorem of Gauss, we have

$$\begin{aligned} |(L,\psi)_\Omega| &= |(\nabla L,g)_\Omega|=|(\nabla\Psi,\nabla L)_\Omega| \\ &\leq|(\mathrm{Div}(\mu D(u))-\nabla \mathrm{div}u,\nabla\Psi)_\Omega| \\ &\quad+|(\nabla\{<\mu D(u)n,n>-(B+\sigma\Delta_\Gamma)h-\mathrm{div}u\},\nabla\Psi)_\Omega| \\ &\leq C_{M_2}\{(\|\nabla u\|_{L_q(\Gamma)}+\|(h,\nabla h,\nabla^2h)\|_{L_q(\Gamma)})\|\nabla\Psi\|_{L_q(\Gamma)}+(\|\nabla u\|_{L_q(\Omega)}+\|h\|_{H_q^2(\Omega)})\|\nabla^2\Psi\|_{L_q(\Omega)}\}. \end{aligned}$$

Using the interpolation inequality: $\|v\|_{L_q(\Gamma)}\leq C\|\nabla v\|^{1/q}_{L_q(\Omega)}\|v\|^{1-1/q}_{L_q(\Omega)}$ and (7.8), we have

$$|(L,\psi)_\Omega|\leq\{\omega(\|\nabla^2u\|_{L_q(\Omega)}+\|h\|_{H_q^3(\Omega)})+C_{\omega,M_2}(\|\nabla u\|_{L_q(\Omega)}+\|h\|_{H_q^2(\Omega)})\}\|\psi\|_{L_{q'}(O)},$$

which leads to

$$\|L\|_{L_q(\Omega)}\leq\omega(\|\nabla^2u\|_{L_q(\Omega)}+\|h\|_{H_q^3(\Omega)})+C_{\omega,M_2}(\|\nabla u\|_{L_q(\Omega)}+\|h\|_{H_q^2(\Omega)}).$$

Thus, we have (7.5).

Putting (7.4) and (7.5) together and choosing ω and M_1 small enough and λ_0 large enough, we have (7.1). This completes the proof of Theorem 7.3.

Maximal L_p - L_q regularity

In this section, we prove Theorem 1.9. As an auxiliary problem, we consider the following equations:

$$\begin{cases} \partial_t u - \mathrm{D}iv(\mu \mathrm{D}(u) - p\mathrm{I}) = \mathrm{F} & \text{in } \Omega \times (0,\mathrm{T}), \\ \mathrm{d}iv\ \mathrm{u} = G = \mathrm{d}iv\mathrm{G} & \text{in } \Omega \times (0,\mathrm{T}), \\ (\mu \mathrm{D}(\mathrm{u}) - p\mathrm{I})\mathrm{n} = \mathrm{H} & \text{on } \Gamma \times (0,\mathrm{T}), \\ \mathrm{u}\,|_{t=0} = u_0 & \text{on } \Omega. \end{cases} \tag{8.1}$$

The corresponding generalized resolvent problem to Eq. (8.1) is

$$\begin{cases} \lambda v - \mathrm{D}iv(\mu \mathrm{D}(\mathrm{v}) - q\mathrm{I}) = f & \text{in } \Omega, \\ \mathrm{d}iv\ \mathrm{v} = g = \mathrm{d}iv g & \text{in } \Omega, \\ (\mu \mathrm{D}(\mathrm{v}) - q\mathrm{I})\mathrm{n} = h & \text{on } \Gamma. \end{cases} \tag{8.2}$$

The following theorem was essentially proved by Shibata[34] and can be proved by using the same argument as in the proof of Theorem 1.7.

Theorem 8.1 Let $1 < q < \infty$ and $0 < \varepsilon < \pi/2$. Assume that the following conditions are satisfied:

i. Ω is a uniformly C^2 domain.

ii. μ is a real valued function satisfying the assumption (1.2).

iii. The weak Dirichlet problem is uniquely solvable on $\hat{H}^1_{q,0}(\Omega)$ and $\hat{H}^1_{q',0}(\Omega)$.

Set

$$X'_q(\Omega) = \{(f, g, \mathrm{g}, \mathrm{h}) \mid \mathrm{f} \in L_q(\Omega)^N, (g, \mathrm{g}) \in \mathrm{D}I_q(\Omega), \mathrm{h} \in H^1_q(\Omega)^N\},$$

$$X_{q'}(\Omega) = \{(F_1, F_3, F_4, F_5, F_6, F_7) \mid F_1, F_3, F_7 \in L_q(\Omega)^N, F_4 \in H^1_q(\Omega)^N, F_5 \in L_q(\Omega), F_6 \in H^1_q(\Omega)^N\}.$$

Then, there exist a constant $\lambda_0 \geq 1$ and operator families $A_0(\lambda)$ and $P_0(\lambda)$ with

$$A_0(\lambda) \in \mathrm{H}ol(\Sigma_{\varepsilon,\lambda_0}, L\,(X\,'(\Omega), H^2_q(\Omega)^N)), \quad P_0(\lambda) \in \mathrm{H}ol(\Sigma_{\varepsilon,\lambda_0}, L\,(X\,'(\Omega), H^1_q(\Omega) + \hat{H}^1_{q,0}(\Omega)))$$

such that for any $\lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\lambda_0}$ and $(f, g, g, h) \in X^1_q(\Omega)$, $v = A_0(\lambda)F_\lambda(f, g, g, h)$ and $q = P_0(\lambda)F_\lambda(f, g, g, h)$ are unique solutions of Eq. (8.2), where $F_\lambda(f, g, g, h) = (f, \lambda^{1/2}g, g, \lambda g, \lambda^{1/2}h, h)$, and

$$\begin{aligned} &R_{L\,(Y_q(\Omega), H_q^{2-j}(\Omega)^N)}(\{(\tau\partial_\tau)^\ell(\lambda^{j/2}A_0(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq r_b, \\ &R_{L\,(Y_q(\Omega), L_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell(\nabla P_0(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\}) \leq r_b \end{aligned} \tag{8.3}$$

for $\ell = 0,1$ and $j = 0,1,2$. Where, r_b is a constant depending on m_0, m_1, ε, q, K, α, β, and N.

Using Theorem 8.1 we shall prove the following theorem.

Theorem 8.2: Let $1 < p, q < \infty$ and $T > 0$. Assume that the conditions i, ii, and iii in Theorem 8.1 are satisfied. Assume that $2/p + 1/q \neq 1$. Then, there exists a $\gamma_0 > 0$ for which the following assertion holds: Let $u_0 \in B^{2(1-1/p)}_{q,p}(\Omega)^N$ be initial data for problem (8.1), and let $\mathrm{F}, G, \mathrm{G}$ and H are functions appearing in the right hand side of (8.1) with

$$F \in L_p((0,T), L_q(\Omega)^N), \quad e^{-\gamma t}G \in L_p(\mathbb{R}, H^1_q(\Omega)) \cap H^1_p(\mathbb{R}, L_q(\Omega)),$$

$$e^{-\gamma t}G \in H^1_p(\mathbb{R}, L_q(\Omega)^N), \quad e^{-\gamma t}H \in L_p(\mathbb{R}, H^1_q(\Omega)^N) \cap H^1_p(\mathbb{R}, L_q(\Omega)^N)$$

for any $\gamma \geq \gamma_0$. Assume that the compatibility conditions:

$$\mathrm{u}_0 - \mathrm{G}\,|_{t=0} \in J_q(\Omega), \quad (\mu \mathrm{D}(\mathrm{u}_0)\mathrm{n})_| tau = (\mathrm{H}\,|_{t=0})_\tau \quad \text{for } 2/\mathrm{p} + 1/\mathrm{N} < 1 \tag{8.4}$$

holds. Then, problem (8.1) admits solutions u and p with

$$u \in L_p((0,T), H^2_q(\Omega)^N) \cap H^1_p((0,T), L_q(\Omega)^N), \quad p \in L_p((0,T), H^1_q(\Omega) + \hat{H}^1_{q,0}(\Omega))$$

possessing the estimate:

$$\|u\|_{L_p((0,T),H_q^2(\Omega))}+\|\partial_t u\|_{L_p((0,T),L_q(\Omega))}+\|\nabla p\|_{L_p((0,T),L_q(\Omega))}$$
$$\leq Ce^{\gamma T}\{\|u_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)}+\|\mathrm{F}\|_{L_p((0,T),L_q(\Omega))}+\|e^{-\gamma t}(G,\mathrm{H})\|_{L_p(R,H_q^1(\Omega))}$$
$$+c_\gamma\|e^{-\gamma t}(G,\mathrm{H})\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))}+\|e^{-\gamma t}\partial_t\mathrm{G}\|_{L_p(\mathbb{R},L_q(\Omega))}\}$$

for any $\gamma\geq\gamma_0$. Where, C is a constant independent of $\gamma\geq\gamma_0$ and $c_\gamma=\sup_{\tau\in R}((\gamma^2+\tau^2)/(1+\tau^2))^{1/4}$.

Proof: Let $\mathbf{F}_0$ be the zero extension of $\mathbf{F}$ outside of $(0,T)$, that is $\mathbf{F}_0(\cdot,t)=\mathbf{F}(\cdot,t)$ for $t\in(0,T)$ and $\mathbf{F}_0(\cdot,t)=0$ for $t\notin(0,T)$. We consider the following problem:

$$\begin{cases}\partial_t \mathrm{u}-\mathrm{Div}(\mu\mathrm{D}(u)-p\mathrm{I})=F_0 & \text{in } \Omega\times\mathrm{R},\\ \mathrm{div}\,\mathrm{u}=G=\mathrm{div}\mathrm{G} & \text{in } \Omega\times\mathrm{R},\\ (\mu\mathrm{D}(\mathrm{u})-p\mathrm{I})\mathrm{n}=\mathrm{H} & \text{on } \Gamma\times\mathrm{R}.\end{cases} \tag{8.5}$$

Let L and L^{-1} be the Laplace transform and the inverse Laplace transform defined by

$$L[f](\lambda)=\hat{f}(\lambda)=\int_{\mathbb{R}}e^{-\lambda t}f(t)dt,\quad L^{-1}[g](t)=\frac{1}{2\pi}\int_{\mathbb{R}}e^{\lambda t}g(\tau)d\tau$$

for $\lambda=\gamma+i\tau\in C$. Notice that

$$L[f](\lambda)=F[e^{-\gamma t}f(t)](\tau),\quad L^{-1}[g](t)=e^{\gamma t}F^{-1}[g](t),$$

where F and F^{-1} denote the one dimensional Fourier transform and inverse Fourier transform. Applying the Laplace transform to (8.5) yields that

$$\begin{cases}\lambda\hat{u}-\mathrm{Div}(\mu\mathrm{D}(\hat{u})-\hat{p}I)=\hat{F}_0 & \text{in } \Omega,\\ \mathrm{div}\hat{u}=\hat{G}=\mathrm{div}\hat{\mathrm{G}} & \text{in } \Omega,\\ (\mu\mathrm{D}(\hat{u})-\hat{p}I)\mathrm{n}=\hat{\mathrm{H}} & \text{on } \Gamma.\end{cases}$$

Thus, in view of Theorem 8.1, setting

$$u=L^{-1}[A_0(\lambda)F_\lambda(\hat{F}_0(\lambda),\hat{G}(\lambda),\hat{\mathrm{G}}(\lambda),\hat{\mathrm{H}}(\lambda))](t),\quad p=L^{-1}[P_0(\lambda)F_\lambda(\hat{\mathrm{F}}_0(\lambda),\hat{G}(\lambda),\hat{\mathrm{G}}(\lambda),\mathrm{H}(\lambda))](t),$$

we see that u and p satisfy Eq. (8.5). Applying the estimate (8.3) together with Weis' operator valued Fourier multiplier theorem gives[2]

$$\|e^{-\gamma t}\mathrm{u}\|_{L_p(\mathbb{R},H_q^2(\Omega))}+\|e^{-\gamma t}\partial_t\mathrm{u}\|_{L_p(\mathbb{R},L_q(\Omega))}+\|e^{-\gamma t}\nabla p\|_{L_p(\mathbb{R},L_q(\Omega))}$$
$$\leq C\{\|e^{-\gamma t}\mathrm{F}_0\|_{L_p(\mathbb{R},L_q(\Omega))}+\|e^{-\gamma t}(G,\mathrm{H})\|_{L_p(\mathbb{R},H_q^1(\Omega))}+\|e^{-\gamma t}\Lambda_\gamma^{1/2}(G,\mathrm{H})\|_{L_p(\mathbb{R},L_q(\Omega))}$$
$$+\|e^{-\gamma t}\partial_t G\|_{L_p(\mathbb{R},L_q(\Omega))}\} \tag{8.6}$$

where we have set $\Lambda_\gamma^{1/2}f=L^{-1}[\lambda^{1/2}\hat{f}(\lambda)]$. By Proposition 3.4, we have

$$\|e^{-\gamma t}\Lambda_\gamma^{1/2}f\|_{L_p(\mathbb{R},L_q(\Omega))}=\|F^{-1}[\lambda^{1/2}(1+\tau^2)^{-1/4}(1+\tau^2)\hat{f}(\lambda)]\|_{L_p(\mathbb{R},L_q(\Omega))}\leq c_\gamma\|e^{-\gamma t}f\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))},$$

which, combined with (8.6), leads to

$$\|u\|_{L_p((0,T),H_q^2(\Omega))}+\|\partial_t u\|_{L_p((0,T),L_q(\Omega))}+\|\nabla p\|_{L_p((0,T),L_q(\Omega))}$$
$$\leq Ce^{\gamma T}\{\|\mathrm{F}\|_{L_p((0,T),L_q(\Omega))}+\|e^{-\gamma t}(G,\mathrm{H})\|_{L_p(R,H_q^1(\Omega))}$$
$$+c_\gamma\|e^{-\gamma t}(\mathrm{G},\mathrm{H})\|_{H_p^{1/2}(\mathbb{R},L_q(\Omega))}+\|e^{-\gamma t}\partial_t\mathrm{G}\|_{L_p(\mathbb{R},L_q(\Omega))}\}. \tag{8.7}$$

To construct a solution of Eq. (8.1), we next consider the initial value problem:

$$\begin{cases}\partial_t v-\mathrm{Div}(\mu\mathrm{D}(\mathrm{v})-q\mathrm{I})=0 & \text{in } \Omega\times(0,\infty),\\ \mathrm{div}\,\mathrm{v}=0 & \text{in } \Omega\times(0,\infty),\\ (\mu D(\mathrm{v})-p\mathrm{I})n=0 & \text{on } \Gamma\times(0,\infty),\\ \mathrm{v}|_{t=0}=v_0 & \text{in } \Omega.\end{cases} \tag{8.8}$$

Where, $\mathrm{v}_0=u_0-\mathrm{u}|_{t=0}$. Obviously, $u+v$ and $p+q$ are required solutions of Eq. (8.1). To solve Eq. (8.8), we formulate it in the semigroup setting. Given v, let $K(v)$ be a unique solution of the weak Dirichlet problem:

$$(\nabla K(v),\nabla\varphi)_\Omega=(\mathrm{Div}(\mu\mathrm{D}(v))-\nabla\mathrm{div}v,\nabla\phi)_\Omega\quad\text{for any}\,\varphi\in\hat{\mathrm{H}}_{q',0}^1(\Omega), \tag{8.9}$$

subject to $K(v) = <\mu D(v)n, n> - \mathrm{div}v$ on Γ. By the assumption iii in Theorem 8.1, we know the unique existence of $K(v) \in H_q^1(\Omega) + \hat{H}_{q,0}^1(\Omega)$ for any $v \in H_q^2(\Omega)$ possessing the estimate:

$$\|\nabla K(\mathrm{v})\|_{L_q(\Omega)} \le C\|\nabla \mathrm{v}\|_{H_q^1(\Omega)}.$$

Instead of (8.8), we consider the equations:

$$\begin{cases} \partial_t v - \mathrm{D}iv(\mu D(\mathrm{v}) - K(\mathrm{v})\mathrm{I}) = f & \text{in } \Omega\times(0,\infty), \\ (\mu \mathrm{D}(\mathrm{v}) - K(\mathrm{v})\mathrm{I})\mathrm{n} = 0 & \text{on } \Gamma\times(0,\infty), \\ \mathrm{v}|_{t=0} = v_0 & \text{in } \Omega \end{cases} \tag{8.10}$$

with $\mathrm{v} \in J_q(\Omega)$ for any $t > 0$. The corresponding resolvent problem to (8.10) is

$$\lambda w - \mathrm{D}iv(\mu \mathrm{D}(\mathrm{w}) - K(\mathrm{w})\mathrm{I}) = f \quad \text{in } \Omega, \quad (\mu \mathrm{D}(w) - \mathrm{K}(w)\mathrm{I})n|_\Gamma = 0.$$

Applying Theorem 8.2, we see that problem (8.8) admits a unique solution $\mathrm{v} \in H_q^2(\Omega)^N$ for any $\lambda \in \Sigma_{\varepsilon,\lambda_0}$ and $\mathrm{f} \in L_q(\Omega)^N$ possessing the estimate:

$$|\lambda|\|\mathrm{w}\|_{L_q(\Omega)} + \|\mathrm{w}\|_{H_q^2(\Omega)} \le C\|\mathrm{f}\|_{L_q(\Omega)}. \tag{8.11}$$

By (8.9), $(\mu D(\mathrm{w}) - K(\mathrm{w})\mathrm{I})\mathrm{n}|_\Gamma = 0$ is equivalent to $(\mu \mathrm{D}(\mathrm{w})\mathrm{n})_\tau|_\Gamma = 0$ provided that $\mathrm{w} \in J_q(\Omega)$. Thus, we define the domain $D_q(\Omega)$ and the operator A_q associated with Eq. (8.11) by setting

$$D_q(\Omega) = \{w \in J_q(\Omega) \cap H_q^2(\Omega)^N \mid (\mu \mathrm{D}(\mathrm{w})n)_\tau = 0 \text{ on } \Gamma\},$$

$$W_q w = \mathrm{D}iv(\mu \mathrm{D}(\mathrm{w}) - K(w)I) \quad \text{for } \mathrm{w} \in D_{\mathrm{q}}(\Omega).$$

Then, the operator A_q generates a C_0 analytic semigroup $\{T(t)\}_{t\ge0}$ on $J_q(\Omega)$ associated with Eq. (8.11). Let $D_{q,p}(\Omega) = (J_q(\Omega), S_q(\Omega))_{1-1/p,p}$, where $(\cdot,\cdot)_{1-1\,p,p}$ is a real interpolation functor, and then, for any $v_0 \in D_{q,p}(\Omega)$, $v(\cdot,t) = T(\mathrm{t})v_0$ is a unique solution of Eq. (8.8) possessing the estimate:

$$\|e^{-\gamma t}\mathrm{v}\|_{L_p((0,\infty),H_q^2(\Omega))} + \|e^{-\gamma t}\partial_t \mathrm{v}\|_{L_p((0,\infty),L_q(\Omega))} \le C\|\mathrm{v}_0\|_{B_{q,p}^{2(1-1/p}(\Omega)} \tag{8.12}$$

for any $\gamma \ge \lambda_0$, where C is independent of γ. Notice that $v \in D_{0q,p}(\Omega)$ holds if and only if $\mathrm{v} \in B_{q,p}^{2(1-1/p)}(\Omega)$ and $(\mu D(\mathrm{v})\mathrm{n})_\tau = 0$ on Γ for $2/p + 1/q < 1$, and $\mathrm{v} \in B_{q,p}^{2(1-1/p)}(\Omega)$ for $2/p + 1/q > 1$. In particular, by (8.4), $\mathrm{u}_0 - \mathrm{u}|_{t=0} \in D_{q,p}(\Omega)$ provided that $2/p + 1/q \ne 1$. Thus, $\mathrm{v} = T(t)(\mathrm{u}_0 - \mathrm{u}|_{t=0})$ is a unique solution of Eq. (8.8) with $\mathrm{v}_0 = \mathrm{u}_0 - \mathrm{u}|_{t=0}$ and by (8.12) we have

$$\|e^{-\gamma t}\mathrm{v}\|_{L_p((0,\infty),H_q^2(\Omega))} + \|e^{-\gamma t}\partial_t \mathrm{v}\|_{L_p((0,\infty),L_q(\Omega))} \le C(\|\mathrm{u}_0\|_{B_{q,p}^{2(1-1/p}(\Omega)} + \|\mathrm{u}|_{t=0}\|_{B_{q,p}^{2(1-1/p}(\Omega)}).$$

By real interpolation, we know that

$$\|\mathrm{u}|_{t=0}\|_{B_{q,p}^{2(1-1/p}(\Omega)} \le C(\|e^{-\gamma t}\mathrm{u}\|_{L_p((0,\infty),H_q^2(\Omega))} + \|e^{-\gamma t}\partial_t \mathrm{u}\|_{L_p((0,\infty),L_q(\Omega))}).$$

Thus, by (8.7)

$$\|\mathrm{v}\|_{L_p((0,T),H_q^2(\Omega))} + \|\partial_t \mathrm{v}\|_{L_p((0,T),L_q(\Omega))}$$
$$\le Ce^{\gamma T}(\|\mathrm{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|\mathrm{F}\|_{L_p((0,T),L_q(\Omega))} + \|e^{-\gamma t}(G,\mathrm{H})\|_{L_p(R,H_q^1(\Omega))}$$
$$+c_\gamma\|e^{-\gamma t}(G,\mathrm{H})\|_{H_p^{1/2}(R,L_q(\Omega))} + \|e^{-\gamma t}\partial_t G\|_{L_p(R,L_q(\Omega))}\}$$

This completes the proof of Theorem 8.2.

We now study the equations:

$$\begin{cases} \partial_t v - \mathrm{D}iv(\mu \mathrm{D}(\mathrm{v}) - p\mathrm{I}) = 0, \quad \mathrm{d}iv\mathrm{v} = 0 & \text{in } \Omega\times(0,\mathrm{T}), \\ \partial_t \rho + A_\sigma \cdot \nabla_\Gamma \rho - \mathrm{v}\cdot\mathrm{n} + F\ \mathrm{v} = F & \text{on } \Gamma\times(0,\mathrm{T}), \\ (\mu \mathrm{D}(v) - q\mathrm{I} - ((B + \delta\Delta_\Gamma)\rho)\mathrm{I})\mathrm{n} = 0 & \text{on } \Gamma\times(0,\mathrm{T}), \\ (v,\rho)|_{t=0} = (0,0) & \text{in } \Omega\times\Gamma. \end{cases} \tag{8.13}$$

We shall prove the following theorem.

Theorem 8.3: let $1 < p,q < \infty$ and $T > 0$. Assume that the conditions i–iv stated in Theorem 1.7 are satisfied. Then, for any $F \in L_p((0,T),W_q^{2-1/q}(\Gamma))$, problem (8.13) admits solutions v and ρ with

$$\mathrm{v} \in L_p((0,T),H_q^2(\Omega)^N) \cap H_p^1((0,T),L_q(\Omega)^N), \quad \rho \in L_p((0,T),H_q^3(\Omega)) \cap H_p^1((0,T),H_q^2(\Omega))$$

possessing the estimate:

$$\|\mathrm{v}\|_{L_p((0,T),H_q^2(\omega))}+\|\partial_t \mathrm{v}\|_{L_p((0,T),L_q(\Omega))}+\|\rho\|_{L_p((0,T),H_q^3(\Omega))}+\|\partial_t\rho\|_{L_p((0,T),H_q^2(\Omega))}$$
$$\le Ce^{c\gamma_\sigma T}\|\mathrm{F}\|_{L_p((0,T),W_q^{2-1/q}(\Gamma))}.$$

Proof: Let F_0 be the zero extension of F outside of $(0,T)$ and we consider the equations:

$$\begin{cases}\partial_t v-Div(\mu\mathrm{D}(v)-\mathfrak{p}\mathrm{I})=0, \quad div v=0 & \text{in } \Omega\times\mathbb{R},\\ \partial_t\rho+A_\sigma\cdot\nabla_\Gamma\rho-\mathrm{v}\cdot\mathrm{n}+F\ v=F_0 & \text{on } \Gamma\times\mathbb{R},\\ (\mu\mathrm{D}(v)-qI-((B+\delta\Delta_\Gamma)\rho)I)n=0 & \text{on } \Gamma\times\mathbb{R}.\end{cases} \tag{8.14}$$

Let $\hat{F}_0$ be the Laplace transform of F_0 and let $A(\lambda)$, $P(\lambda)$ and $H(\lambda)$ be the operators given in Theorem 1.7. Then, v and ρ are given by

$$\mathrm{v}=L^{-1}[A(\lambda)E\hat{F}_0(\lambda)],\quad \mathfrak{p}=L^{-1}[P(\lambda)E\hat{F}_0(\lambda)],\quad \rho=L^{-1}[H(\lambda)E\hat{F}_0(\lambda)],$$

where $E\hat{F}_0(\lambda)=(0,\hat{F}_0(\lambda),0,0,0,0,0)$. Applying Theorem 1.7 together with Weis' oprator valued Fourier multiplier theorem gives[2]

$$\mathrm{v}\in L_{p,loc}(\mathbb{R},H_q^2(\Omega)^N)\cap H^1_{p,loc}(\mathbb{R},L_q(\Omega)^N),\quad \mathfrak{p}\in L_{p,loc}(\mathbb{R},H_q^1(\Omega)+\hat{H}^1_{q,0}(\Omega)),$$
$$\rho\in L_{p,loc}(\mathbb{R},H_q^3(\Omega))\cap H^1_{p,loc}(\mathbb{R},H_q^2(\Omega)),$$

and

$$\|e^{-\gamma t}\mathrm{v}\|_{L_p(\mathbb{R},H_q^2(\Omega))}+\|e^{-\gamma t}\partial_t \mathrm{v}\|_{L_p(\mathbb{R},L_q(\Omega))}+\|e^{-\gamma t}\rho\|_{L_p(\mathbb{R},H_q^3(\Omega))}+\|e^{-\gamma t}\partial_t\rho\|_{L_p(\mathbb{R},H_q^2(\Omega))}$$
$$\le C\|e^{-\gamma t}\mathrm{F}_0\|_{L_p(\mathbb{R},W_q^{2-1/q}(\Gamma))}\le C\|\mathrm{F}\|_{L_p((0,T),W_q^{2-1/q}(\Gamma))} \tag{8.15}$$

for any $\gamma\ge c\gamma_\sigma$. Since $|\gamma/\lambda|\le 1$ for $\lambda=\gamma+i\tau\in C$, we have

$$\begin{aligned}\gamma\|e^{-\gamma t}\mathrm{v}\|_{L_p(\mathbb{R},L_q(\Omega))}&\le C\|e^{-\gamma t}\partial_t\mathrm{v}\|_{L_p(\mathbb{R},L_q(\Omega))},\\ \gamma\|e^{-\gamma t}\rho\|_{L_p(\mathbb{R},W_q^{2-1/q}(\Gamma))}&\le C\|e^{-\gamma t}\partial_t\rho\|_{L_p(\mathbb{R},W_q^{1-1/q}(\Gamma))}.\end{aligned} \tag{8.16}$$

Combining (8.15) and (8.16) gives

$$\|\mathrm{v}\|_{L_p((-\infty,0),L_q(\Omega))}+\|\rho\|_{L_p((-\infty,0),H_q^2(\Omega))}\le\|e^{-\gamma t}\mathrm{v}\|_{L_p(\mathbb{R},L_q(\Omega))}+\|e^{-\gamma t}\rho\|_{L_p(\mathbb{R},H_q^2(\Omega))}$$
$$\le\gamma^{-1}(\|e^{-\gamma t}\partial_t\mathrm{v}\|_{L_p(\mathbb{R},L_q(\Omega))}+\|e^{-\gamma t}\partial_t\rho\|_{L_p(\mathbb{R},H_q^2(\Omega))})\le C\gamma^{-1}\|\mathrm{F}\|_{L_p((0,T),W_q^{2-1/q}(\Gamma))}.$$

Letting $\gamma\to\infty$, wee see that $\|\mathrm{v}\|_{L_p((-\infty,0),L_q(\Omega))}=\|\rho\|_{L_p((-\infty,0),H_q^2(\Omega))}=0$, which leads to $(\mathrm{v},\rho)|_{t=0}=(0,0)$. This completes the proof of Theorem 8.3.

We finally study the initial problem:

$$\begin{cases}\partial_t v-Div(\mu\mathrm{D}(\mathrm{v})-\mathfrak{p}\mathrm{I})=0, \quad div\mathrm{v}=0 & \text{in } \Omega\times(0,\mathrm{T}),\\ \partial_t\rho+A_\sigma\cdot\nabla_\Gamma\rho-\mathrm{v}\cdot\mathrm{n}+F\ v=0 & \text{on } \Gamma\times(0,\mathrm{T}),\\ (\mu\mathrm{D}(\mathrm{v})-\mathfrak{p}\mathrm{I}-((B+\delta\Delta_\Gamma)\rho)\mathrm{I})\mathrm{n}=0 & \text{on } \Gamma\times(0,\mathrm{T}),\\ (\mathrm{v},\rho)|_{t=0}=(0,\rho_0) & \text{in } \Omega\times\Gamma.\end{cases} \tag{8.17}$$

We shall prove the following theorem.

Theorem 8.4: let $1<p,q<\infty$. Assume that the conditions ii–iv stated in Theorem 1.7 are satisfied and that Ω is a uniformly C^3 domain whose inside is finitely covering. Then, for any $\rho_0\in B^{3-1/p-1/q}_{q,p}(\Gamma)$, problem (8.17) admits unique solutions v, $\mathfrak{p}$, and ρ with

$$\mathrm{v}\in L_{p,loc}((0,\infty),H_q^2(\Omega)^N)\cap H^1_{p,loc}((0,\infty),L_q(\Omega)^N),\quad \mathfrak{p}\in L_{p,loc}((0,\infty),H_q^1(\Omega)+\hat{H}^1_{q,0}(\Omega)),$$
$$\rho\in L_{p,loc}((0,\infty),H_q^3(\Omega))\cap H^1_{p,loc}((0,\infty),H_q^2(\Omega))$$

possessing the estimate:

$$\|\mathrm{v}\|_{L_p((0,T),H_q^2(\Omega))}+\|\partial_t\mathrm{v}\|_{L_p((0,T),L_q(\Omega))}+\|\rho\|_{L_p((0,T),H_q^3(\Omega))}+\|\partial_t\rho\|_{L_p((0,T),H_q^2(\Omega))}$$
$$\le Ce^{\gamma T}\gamma_\sigma\|\rho_0\|_{B^{3-1/p-1/q}_{q,p}(\Gamma)}.$$

for any $T\in(0,\infty)$ and $\gamma\ge c\gamma_\sigma$.

Proof: First we consider the case where $\sigma = 0$. The corresponding resolvent problem to Eq. (8.17) is

$$\begin{cases} \lambda u - \mathrm{D}iv(\mu \mathrm{D}(\mathrm{u}) - K(\mathrm{u},h)I) = f & \text{in } \Omega, \\ \lambda h + A_\sigma \cdot \nabla_\Gamma h - \mathrm{u}\cdot \mathrm{n} + F\ \mathrm{u} = \mathrm{g} & \text{on } \Gamma, \\ (\mu \mathrm{D}(\mathrm{v})\mathrm{n})_\tau = 0, \quad \mathrm{d}iv\mathrm{u} = 0 & \text{on } \Gamma. \end{cases} \tag{8.18}$$

Where, $K(u,h)$ is a unique solution of the weak Dirichlet problem:

$$(\nabla K(\mathrm{u},h), \nabla \varphi)_\Omega = (\mathrm{D}iv(\mu \mathrm{D}(\mathrm{u})) - \nabla \mathrm{d}iv\mathrm{u}, \nabla \varphi)_\Omega \quad \text{for any} \varphi \in \hat{\mathrm{H}}^1_{\mathrm{q}',0}(\Omega),$$

subject to $K(\mathrm{u,h}) =< \mu \mathrm{D}(\mathrm{u})\mathrm{n},\mathrm{n} > -(B + \delta\Delta_\Gamma)\mathrm{h} - \mathrm{d}iv\mathrm{u}$ on Γ. We know the existence of $K(\mathrm{u},h) \in H^1_q(\Omega) + \hat{H}^1_{q,0}(\Omega)$ for $\mathrm{u} \in H^2_q(\Omega)^N$ and $h \in W^{3-1/q}_q(\Gamma)$ possessing the estimate:

$$\|\nabla K(\mathrm{u,h})\|_{L_q(\Omega)} \le C(\|\nabla \mathrm{u}\|_{H^1_q(\Omega)} + \|\mathrm{h}\|_{W^{3-1/q}_q(\Gamma)}).$$

Let

$$H_q = \{(u,h) \mid u \in J_q(\Omega), h \in W^{2-1/q}_q(\Gamma)\},$$

$$D_q = \{(\mathrm{u},h) \in H_q(\Omega) \mid \mathrm{u} \in H^2_q(\Omega), h \in W^{3-1/q}_q(\Gamma), (\mu \mathrm{D}(\mathrm{u})\mathrm{n})_\tau |_\Gamma = 0\},$$

$$A_q(\mathrm{u},h) = (\mathrm{D}iv(\mu \mathrm{D}(\mathrm{u}) - K(\mathrm{u},h)I), (\mathrm{u.n} - F\ u)|_\Gamma) \quad \text{for} (\mathrm{u,h}) \in D_\mathrm{q}.$$

And then, problem (8.18) is written as

$$\lambda(u,h) - A_q(u,h) = (f,g) \quad \text{in } \Omega \times \Gamma. \tag{8.19}$$

And also, the corresponding evolution equation is written as

$$\partial_t(v,\rho) - A_q(v,\rho) = (0,0) \quad \text{for } t>0, \quad (\mathrm{v},\rho)|_{t=0} = (u_0,\rho_0) \tag{8.20}$$

with $p = K(\mathrm{v},\rho)$ and $\mathrm{u}_0 = 0$, where $\mathrm{v} \in D_q$ for $t > 0$. By Theorem 1.7 and Theorem 1.11, we see that there exists a $\lambda_0 > 0$ such that for any $\lambda \in \Sigma_{\varepsilon,\lambda_0}$ and $(f,g) \in H_q(\Omega)$, problem (8.19) admits a unique solution $(\mathrm{u},h) \in D_q$ possessing the estimate:

$$|\lambda| \|(\mathrm{u},h)\|_{H_q} + \|(\mathrm{u},h)\|_{D_q} \le C\|(\mathrm{f},g)\|_{H_q}.$$

Where, we have set

$$\|(\mathrm{u},h)\|_{H_q} = \|\mathrm{u}\|_{L_q(\Omega)} + \|h\|_{W^{2-1/q}_q(\Gamma)}, \quad \|(u,h)\|_{D_q} = \|\mathrm{u}\|_{H^2_q(\Omega)} + \|h\|_{W^{3-1/q}_q(\Gamma)}.$$

Thus, A_q generates a C_0 analytic semigroup $\{T(t)\}_{t\ge 0}$ associated with Eq. (8.20) possessing the estimate:

$$\|T(t)(\mathrm{f},g)\|_{H_q(\Omega)} \le Ce^{\lambda_0 t}\|(\mathrm{f},g)\|_{H_q(\Gamma)}$$

for any $t > 0$. Let $D_{q,p} = (H_q, D_q)_{1-1/p,p}$, where $(\cdot,\cdot)_{1-1/p,p}$ is a real interpolation functor. By real interpolation method, we see that for any $(u_0,\rho_0) \in D_{q,p}$, problem (8.20) admits a unique solution $(\mathrm{v},\rho) = T(t)(\mathrm{u}_0,\rho_0)$ possessing the estimate:

$$\|e^{-\gamma t}\mathrm{v}\|_{L_p((0,\infty),H^2_q(\Omega))} + \|e^{-\gamma t}\partial_t \mathrm{v}\|_{L_p((0,\infty),L_q(\Omega))} + \|e^{-\gamma t}\rho\|_{L_p((0,\infty),W^{3-1/q}_q(\Gamma))}$$
$$+ \|e^{-\gamma t}\partial_t\rho\|_{L_p((0,\infty),W^{2-1/q}_q(\Gamma))} \le C(\|u_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|\rho_0\|_{W^{3-1/p-1/q}_{q,p}(\Gamma)}).$$

Since Ω is a uniform C^3 domain, we can construct an extension, $\tilde{\rho}$, of ρ to Ω such that $\tilde{\rho}|_\Gamma = \rho$, $\tilde{\rho} \in L_{p,\mathrm{loc}}((0,\infty),H^3_q(\Omega)) \cap H^1_p((0,\infty),H^2_q(\Omega))$, and

$$\|\tilde{\rho}\|_{L_p((0,T),H^3_q(\Omega))} \le C\|\rho\|_{L_p((0,T),W^{3-1/q}_q(\Gamma))},$$

$$\|\partial_t\tilde{\rho}\|_{L_p((0,T),H^2_q(\Omega))} \le C\|\partial_t\rho\|_{L_p((0,T),W^{2-1/q}_q(\Gamma))},$$

where C is a constant independent of T. We write $\tilde{\rho}$ simply by ρ. Since $(0,\rho_0) \in D_{q,p}$ for $\rho_0 \in B^{3-1/p-1/q}_{q,p}(\Gamma)$, we then have Theorem 8.4 in the case where $\sigma = 0$.

We next consider the case where $\sigma \in (0,1)$. Let (v_1, p_1, h_1) be a solution of Eq. (8.17) in the case where $\sigma = 0$ possessing the estimate:

$$\|\mathrm{v}_1\|_{L_p((0,T),H^2_q(\Omega))} + \|\partial_t\mathrm{v}_1\|_{L_p((0,T),L_q(\Omega))} + \|\rho_1\|_{L_p((0,\infty),H^3_q(\Omega))}$$
$$+ \|\partial_t\rho_1\|_{L_p((0,\infty),H^2_q(\Omega))} \le Ce^{\gamma \mathrm{T}}\|\rho_0\|_{W^{3-1/p-1/q}_{q,p}(\Gamma)}$$

for any $\gamma \geq \lambda_0$, where C is independent of $\gamma \geq \lambda_0$. Let (v_2, p_2, ρ_2) be a unique solution of the equations:

$$\begin{cases} \partial_t v_2 - \mathrm{D}iv(\mu \mathrm{D}(v_2) - p_2 \mathrm{I}) = 0, \quad \mathrm{d}iv v_2 = 0 & \text{in } \Omega \times (0,T), \\ \partial_t \rho_2 + A_\sigma \cdot \nabla_\Gamma \rho_2 - v_2 \cdot n + F\ v_2 = -A_\sigma \cdot \nabla_\Gamma \rho_1 & \text{on } \Gamma \times (0,T), \\ (\mu D(v_2) - p_2 \mathrm{I} - ((B + \delta \Delta_\Gamma)\rho_2)\mathrm{I})n = 0 & \text{on } \Gamma \times (0,T), \\ (v_2, \rho_2)|_{t=0} = (0,0) & \text{in } \Omega \times \Gamma. \end{cases}$$

By Theorem 8.3 and (1.3), we know the existence of v_2, p_2 and ρ_2 with

$$v_2 \in H_p^1((0,T), L_q(\Omega)^N) \cap L_p((0,T), H_q^2(\Omega)^N), \quad p_2 \in L_p((0,T), H_q^1(\Omega) + H_{q,0}^1(\Omega)),$$
$$\rho_2 \in H_{p,loc}^1((0,\infty), H_q^2(\Omega)) \cap L_{p,loc}((0,\infty), H_q^3(\Omega))$$

possessing the estimate:

$$\|v_2\|_{L_p((0,T),H_q^2(\Omega))} + \|\partial_t v_2\|_{L_p((0,T),L_q(\Omega))} + \|\rho_2\|_{L_p((0,\infty),H_q^3(\Omega))} + \|\partial_t \rho_2\|_{L_p((0,\infty),H_q^2(\Omega))}$$
$$\leq Ce^{c\gamma_\sigma T} \|A_\sigma \cdot \rho_1\|_{L_p((0,T),W_q^{2-1/q}(\Gamma))} \leq Ce^{c\gamma_\sigma T} \sigma^{-b} \|\rho_1\|_{L_p((0,T),W_q^{3-1/q}(\Gamma))}$$
$$\leq Ce^{(c\gamma_\sigma + \gamma)T} \sigma^{-b} \|\rho_0\|_{B_{q,p}^{3-1/p-1/q}(\Gamma)}$$

for any $\gamma \geq \lambda_0$. Without loss of generality, we may assume that $\lambda_0 \leq c\sigma^{-b}$, because $\sigma \in (0,1)$ and $b > 0$. Thus, setting $v = v_1 + v_2$, $p = p_1 + p_2$ and $\rho = \rho_1 + \rho_2$, we see that v, p and ρ are required solutions of Eq. (8.17).

Applying Theorem 1.11 to the Laplace transform of solutions of the homogeneous equations, we have the uniqueness of solutions of Eq. (8.17). This completes the proof of Theorem 8.4.

Proof of Existence part, theorem 1.9: Let v_1, p_1 be solutions of Eq. (8.1), let v_2, p_2, ρ_2 be solutions of Eq. (8.13) with $F = \mathrm{D} + v_2 \cdot n - F\ v_2$, and let v_3, p_3 and ρ_3 be solutions of Eq. (8.17). Setting $v = v_1 + v_2 + v_3$, $p = p_1 + p_2 + p_3$ and $\rho = \rho_2 + \rho_3$, we see that v, p, and ρ are required solutions of Eq. (1.6).

Proof of Uniqueness, theorem 1.11: Let v, p, and ρ be solutions of the homogeneous equations:

$$\begin{cases} \partial_t v - \mathrm{D}iv(\mu \mathrm{D}(v) - p\mathrm{I}) = 0, \quad \mathrm{d}iv v = 0 & \text{in } \Omega \times (0,T), \\ \partial_t \rho + A_\sigma \cdot \nabla_\Gamma \rho - v \cdot n + F\ v = 0 & \text{on } \Gamma \times (0,T), \\ (\mu D(v) - p\mathrm{I} - ((B + \delta \Delta_\Gamma)\rho)\mathrm{I})n = 0 & \text{on } \Gamma \times (0,T), \\ (v, \rho)|_{t=0} = (0,0) & \text{in } \Omega \times \Gamma, \end{cases} \tag{8.21}$$

with

$$v \in L_p((0,T), H_q^2(\Omega)^N) \cap H_p^1((0,T), L_q(\Omega)^N), \quad p \in L_p((0,T), H_q^1(\Omega) + \hat{H}_q^1(\Omega)),$$
$$\rho \in L_p((0,T), H_q^3(\Omega)) \cap H_p^1((0,T), H_q^2(\Omega)).$$

For any f defined on $(0,T)$, let $E[f]$ be an extension of f outside of $(0,T)$ defined by setting

$$E[f](t) = \begin{cases} 0 & \text{for} \quad t<0, \\ f(\cdot,t) & \text{for} \quad t \in (0,T), \\ f(\cdot, 2T - t) & \text{for} \quad t \in (T,2T), \\ 0 & \text{for} \quad t>2T. \end{cases}$$

If $f|_{t=0}$, then

$$\partial_t E[f](t) = \begin{cases} 0 & \text{for} \quad t<0, \\ (\partial_t f)(\cdot,t) & \text{for} \quad t \in (0,T), \\ -(\partial_t f)(\cdot, 2T - t) & \text{for} \quad t \in (T,2T), \\ 0 & \text{for} \quad t>2T. \end{cases}$$

Thus,

$$E[v] \in H_p^1(\mathbb{R}, L_q(\Omega)^N) \cap L_p(\mathbb{R}, H_q^2(\Omega)^N), \quad E[p] \in L_p(\mathbb{R}, H_q^1(\Omega) + \hat{H}_{q,0}^1(\Omega)),$$
$$E[\rho] \in H_p^1(\mathbb{R}, H_q^2(\Omega)) \cap L_p(\mathbb{R}, H_q^3(\Omega)).$$

Moreover, by Eq. (8.21), we see that

$$\begin{cases} \partial_t E[v] - \mathrm{Div}(\mu \mathrm{D}(E[\mathrm{v}]) - E[p]\mathrm{I}) = 0, \quad \mathrm{div}\, E[\mathrm{u}] = 0 & \text{in } \Omega \times \mathbb{R}, \\ \partial_t E[\rho] + A_\sigma \cdot \nabla_\Gamma E[\rho] - E[\mathrm{v}] \cdot \mathrm{n} + \mathcal{F}\,[\mathrm{v}] = 0 & \text{on } \Gamma \times \mathbb{R}, \\ (\mu \mathrm{D}(E[\mathrm{v}]) - [p]\mathrm{I} - ((\mathcal{B} + \delta\Delta_\Gamma)[\rho])\mathrm{I})\mathrm{n} = 0 & \text{on } \Gamma \times \mathbb{R}. \end{cases} \tag{8.22}$$

Let $u = \mathcal{L}\,[E[\mathrm{v}]](\lambda)$, $q = \mathcal{L}\,[E[p]](\lambda)$, and $h = \mathcal{L}\,[E[\rho]](\lambda)$. Since $\mathrm{E[v]}$, $E[p]$ and $E[\rho]$ vanish outside of $(0,T)$, u, q, and h are entire functions. By Hölder's inequality, we have

$$\|\mathrm{u}\|_{H_q^2(\Omega)} \le \int_0^T \left\|e^{-\lambda t} v(\cdot,t)\right\|_{H_q^2(\Omega)} \le \left(\int_0^T e^{-(\mathrm{Re}\,\lambda)tp'}\,dt\right)^{1/p'} \|\mathrm{v}\|_{L_p((0,T),H_q^2(\Omega))} < \infty,$$

and so $\mathrm{u} \in H_q^2(\Omega)^N$. In the same way, we see that $q \in H_q^1(\Omega) + \hat{H}_{q,0}^1(\Omega)$ and $h \in H_q^3(\Omega)$. Morevoer, applying the Laplace transform to Eq. (8.21), u, q and h satisfy the homogeneous equations:

$$2\lambda v - \mathrm{Div}(\mu D(v) - qI) = 0, \quad v = 0 \quad \text{in } \Omega,$$
$$\lambda h + A_\sigma \cdot \nabla_\Gamma h - \mathrm{u} \cdot \mathrm{n} + \mathcal{F}\, u = 0 \quad \text{on } \Gamma,$$
$$(\mu \mathrm{D}(\mathrm{u}) - qI - ((\mathcal{B} + \delta\Delta_\Gamma)h)\mathrm{I})n = 0 \quad \text{on } \Gamma$$

for any $\lambda \in C$. Thus, the uniqueness of the resolvent problem yields that $u = 0$, $q = 0$ and $h = 0$. Thus, applying the inverse Laplace transform, we have $E[\mathrm{v}] = 0$, $E[p] = 0$ and $E[\rho] = 0$, which implies that $u = 0$, $p = 0$ and $\rho = 0$. This completes the proof of Theorem 1.11 2.

On the weak Dirichlet problem in $\mathbb{R}^{\mathrm{N}}$ and $\mathbb{R}_+^{\mathrm{N}}$

In this appendix, we prove the unique existence and regularity theorem for the weak Dirichlet problem in the model cases.

The $\mathbb{R}^N$ case

In this subsection, we consider the following weak Laplace problem in $\mathbb{R}^N$:

$$(\nabla u, \nabla \varphi)_{\mathbb{R}^N} = (f, \nabla \varphi)_{\mathbb{R}^N} \quad \text{for any}\, \varphi \in \hat{\mathrm{H}}_{\mathrm{q}'}^1(\mathbb{R}^{\mathrm{N}}). \tag{9.1}$$

We shall prove the following theorem.

Theorem 9.1 Let $1 < q < \infty$. Then, for any $f \in L_q(\mathbb{R}^N)^N$, the weak Laplace problem (9.1) admits a unique solution $u \in \hat{H}_q^1(\mathbb{R}^N)$ possessing the estimate: $\|\nabla u\|_{L_q(\mathbb{R}^N)} \le C\|f\|_{L_q(\mathbb{R}^N)}$.

Moreover, if we assume that $\mathrm{div}\, f \in L_q(\mathbb{R}^N)$ in addition, then $\nabla^2 u \in L_q(\mathbb{R}^N)^{N^2}$ and

$$\left\|\nabla^2 \mathrm{u}\right\|_{L_q(\mathbb{R}^N)} \le C\|\mathrm{div}\,\mathrm{f}\|_{L_q(\mathbb{R}^N)}.$$

Proof: To prove the theorem, we consider the strong Laplace equation:

$$\Delta u = \mathrm{div}\,\mathrm{f} \quad \text{in } \mathbb{R}^{\mathrm{N}}. \tag{9.2}$$

Let

$$H_{q,\mathrm{div}}^1(D) = \{f \in L_q(\mathrm{D})^N \mid \mathrm{div}\, f \in L_q(\mathrm{D})\},$$

where D is any domain in $\mathbb{R}^N$. Since $C_0^\infty(\mathbb{R}^N)^N$ is dense both in $L_q(\mathbb{R}^N)^N$ and $H_{q,\mathrm{div}}^1(\mathbb{R}^N)$, and so we may assume that $f \in C_0^\infty(\mathbb{R}^N)^N$. Let $\mathcal{F}\,[\mathrm{f}] = \hat{f}$ and $\mathcal{F}^{-1}$ denote the Fourier transfom f and the Fourier inverse transform, respectively. We then set

$$u = -\mathcal{F}^{-1}\left[\frac{\mathcal{F}\,[\mathrm{div}\,\mathrm{f}](\xi)}{|\xi^2|}\right] = -\mathcal{F}^{-1}\left[\frac{\sum_{j=1}^N i\xi_j \mathcal{F}\,[f_j](\xi)}{|\xi|^2}\right]$$

where we have set $f = (f_1, \ldots, f_N)^{\mathrm{T}}$. By the Fourier multiplieara theorem we have

$$\begin{aligned} \|\nabla \mathrm{u}\|_{L_q(\mathbb{R}^N)} &\le C\|\mathrm{f}\|_{L_q(\mathbb{R}^N)}, \\ \left\|\nabla^2 \mathrm{u}\right\|_{L_q(\mathbb{R}^N)} &\le C\|\mathrm{div}\,\mathrm{f}\|_{L_q(\mathbb{R}^N)}. \end{aligned} \tag{9.3}$$

Of course, u satisfies Eq.(9.2).

We now prove that u satisfies the weak Laplace equation (9.2). For this purpose, we use the following lemma.

Lemma 9.2 Let $1<q<\infty$ and let

$$d_q(x)=\begin{cases}(1+|x|^2)^{1/2} & \text{for N}\neq\text{q},\\ (1+|x|^2)^{1/2}\log(2+|x|^2)^{1/2} & \text{for N=q}.\end{cases}$$

Then, for any $\varphi\in\hat{H}^1_q(\mathbb{R}^N)$, there exists a constant c for which

$$\left\|\frac{\varphi-c}{d_q}\right\|_{L_q(\mathbb{R}^N)}\leq C\|\nabla\varphi\|_{L_q\mathbb{R}^N)}$$

with some constant independent of φ and c.

Proof: For a proof, see Galdi [Chapter II].[44] To use Lemma 9.2, we use a cut-off function, ψ_R, of Sobolev's type defined as follows: Let ψ be a function in $C^\infty(\mathbb{R})$ such that $\psi(t)=1$ for $|t|\leq 1/2$ and $\psi(t)=0$ for $|t|\geq 1$, and set

$$\psi_R(x)=\psi(\frac{\ln\ln|x|}{\ln\ln R}).$$

Notice that

$$|\nabla\psi_R(x)|\leq\frac{c}{\ln\,\ln R}\frac{1}{|x|\ln|x|},\quad supp\nabla\psi_R\subset D_R, \tag{9.4}$$

where we have set $D_R=\{x\in\mathbb{R}^N\mid e^{\sqrt{\ln R}}\leq|x|\leq R\}$. Noting that $f\in C_0^\infty(\mathbb{R}^N)^N$, by (9.2) for large $R>0$ and $\varphi\in\hat{H}^1_{q'}(\mathbb{R}^N)$ we have

$$\begin{aligned}(\mathrm{f},\nabla\varphi)_{\mathbb{R}^N}&=(\mathrm{f},\nabla(\varphi-c))_{\mathbb{R}^N}=-(\mathrm{div}\,\mathrm{f},\varphi-c)_{\mathbb{R}^N}=-(\psi_R\mathrm{div}\,\mathrm{f},\varphi-c)_{\mathbb{R}^N}=-(\psi_R\Delta u,\varphi-c)_{\mathbb{R}^N}\\&=((\nabla\psi\mathbb{R})\cdot(\nabla u),\varphi-c)_{\mathbb{R}^N}+(\psi_{\mathbb{R}}\nabla u,\nabla\varphi)_{\mathbb{R}^N},\end{aligned} \tag{9.5}$$

where c is a constant for which

$$\left\|\frac{\varphi-c}{d_{q'}}\right\|_{L_{q'}(\mathbb{R}^N)}\leq C\|\nabla\varphi\|_{L_{q'}(\mathbb{R}^N)}. \tag{9.6}$$

By (9.4) and (9.6), we have

$$\begin{aligned}|((\nabla\psi_{\mathbb{R}})\cdot(\nabla \mathrm{u}),\phi-c)_{\mathbb{R}^N}|&\leq\|d_{q'}(\nabla\psi_{\mathbb{R}})\cdot(\nabla\mathrm{u})\|_{L_q(\mathbb{R}^N)}\left\|\frac{\phi-c}{d_{q'}}\right\|_{L_{q'}(\mathbb{R}^N)}\\&\leq\frac{C}{\ln\,\ln\mathbb{R}}\|\nabla\mathrm{u}\|_{L_q(D_{\mathbb{R}})}\|\nabla\varphi\|_{L_{q'}(\mathbb{R}^N)}\to 0\end{aligned} \tag{9.7}$$

as $R\to\infty$. By (9.5) and (9.7) we see that u satisfies the weak Dirichlet problem (9.1). The uniqueness follows from the existence theorem just proved for the dual problem. Moreover, if $\mathrm{div}\,f\in L_q(\mathbb{R}^N)$ in addition, then $\nabla^2\mathrm{u}\in L_q(\mathbb{R}^N)^{N^2}$, and so by (9.3) we complete the proof of Theorem 9.1.

The half space case

In this subsection, we consider the following weak Dirichlet problem in $\mathbb{R}^N_+$:

$$(\nabla\mathrm{u},\nabla\varphi)_{\mathbb{R}^N_+}=(f,\nabla\varphi)_{\mathbb{R}^N_+}\quad\text{for any}\,\varphi\in\hat{\mathrm{H}}^1_{\mathrm{q}',0}(\mathbb{R}^{\mathrm{N}}_+). \tag{9.8}$$

We shall prove the following theorem.

Theorem 9.3 Let $1<q<\infty$. Then, for any $\mathrm{f}\in L_q(\mathbb{R}^N_+)^N$, the weak Laplace problem (9.8) admits a unique solution $\mathrm{u}\in\hat{H}^1_q(\mathbb{R}^N_+)$ possessing the estimate: $\|\nabla u\|_{L_q(\mathbb{R}^N_+)}\leq C\|f\|_{L_q(\mathbb{R}^N_+)}$.

Moreover, if we assume that $\mathrm{div}\,f\in L_q(\mathbb{R}^N_+)$ in addition, then $\nabla^2\mathrm{u}\in L_q(\mathbb{R}^N_+)^{N^2}$ and

$$\|\nabla^2\mathrm{u}\|_{L_q(\mathbb{R}^N_+)}\leq C\|\mathrm{div}\mathrm{f}\|_{L_q(\mathbb{R}^N_+)}.$$

Proof: We may assume that $\mathrm{f}=(f_1,\ldots,f_N)^{\hat{O}}\in C_0^\infty(\mathbb{R}^N_+)^N$ in the following, because $C_0^\infty(\mathbb{R}^N_+)$ is dense both in $L_q(\mathbb{R}^N_+)^N$ and $H^1_{q,\mathrm{div}}(\mathbb{R}^N_+)$. We first consider the strong Dirichlet problem:

$$\Delta\mathrm{u}=\mathrm{div}\,f\quad\text{in}\,\mathbb{R}^{\mathrm{N}}_+,\quad u|_{x_N=0}=0. \tag{9.9}$$

For any function, $f(x)$, defined in $\mathbb{R}^N_+$, let f^e and f^o be the even extension and the odd extension of f defined in (4.40). Noting that $(\mathrm{div}\mathrm{f})^o=\sum_{j=1}^{N-1}\partial_j(f_j)^o+\partial_N(f_N)^e$, we define u by letting

$$u = -\mathcal{F}^{-1}\left[\frac{\mathcal{F}[(\mathrm{div} f)^o](\xi)}{|\xi|^2}\right] = -\mathcal{F}^{-1}\left[\frac{\sum_{j-1}^{N-1} i\xi_j \mathcal{F}[(f_j)^o](\xi) + i\xi_N \mathcal{F}[(f_N)^e](\xi)}{|\xi|^2}\right].$$

We then have

$$\|\nabla u\|_{L_q(\mathbb{R}^N)} \le C\|f\|L_q(\mathbb{R}^N_+), \quad \|\nabla^2 \mathrm{u}\|_{L_q(\mathbb{R}^N)} \le C\|\mathrm{div} f\|_{L_q(\mathbb{R}^N_+)}, \tag{9.10}$$

and moreover u satisfies Eq.(9.9).

We next prove that u satisfies the weak Dirichlet problem Eq.(9.8). For this purpose, instead of lemma 9.2, we use the Hardy type inequality:

$$\left(\int_0^\infty \left(\int_0^x f(y)dy\right)^p x^{-r-1}dy\right)^{1/p} \le (p/r)\left(\int_0^\infty (yf(y))^p y^{-r-1}dy\right)^{1/p}, \tag{9.11}$$

where $f \ge 0$, $p \ge 1$ and $r > 0$ (cf. Stein [A.4 p.272]).[45] Of course, using zero extension of f suitably, we can replace the interval $(0,\infty)$ by (a,b) for any $0 \le a < b < \infty$ in (9.11). Let $D_{R,2R} = \{x \in \mathbb{R}^N \mid R \le |x| \le 2R\}$. Using (9.11), we see that for any $\varphi \in \hat{H}^1_{q',0}(\mathbb{R}^N_+)$

$$\lim_{R\to\infty} R^{-1}\|\varphi\|_{L_{q'}(D_{R,2R})} = 0. \tag{9.12}$$

In fact, using $\varphi|_{x_N=0} = 0$, we write $\varphi(x', x_N) = \int_0^{x_N} (\partial_s\varphi)(x', s)ds$. Thus, by (9.11) we have

$$\int_a^b |\varphi(x', x_N)|^{q'} dx_N \le \left(\frac{bq'}{q'-1}\right)^{q'} \int_a^b |(\partial_N\varphi)(x', x_N)|^{q'} dx_N$$

for any $0 < a < b$. Let

$$E_R^1 = \{x \in \mathbb{R}^N \,\|\, x'| \le 2R,\ R/2 \le x_N < 2R\}, \quad E_R^2 = \{x \in \mathbb{R}^N \mid 0 \le x_N \le 2R, R/2 \le |x'| \le 2R\},$$

and then $D_{R,2R} \subset E_R^1 \cup E_R^2$. Thus, by (9.11),

$$\left(\int_{D_{R,2R}} |\varphi(x)|^{q'} dx\right)^{1/q'}$$

$$\le \left(\frac{Rq'}{q'-1}\right)\left\{\int_{|x'|\le R}\int_{R/2}^{2R} |\partial_N\varphi(x)|^{q'} dx_N dx' + \int_{R/2\le|x'|\le 2R}\int_0^{2R} |\partial_N\varphi(x)|^{q'} dx_N dx'\right\}^{1/q'},$$

which leads to (9.12).

Let ω be a function in $C_0^\infty(\mathbb{R}^N)$ such that $\omega(x) = 1$ for $|x| \le 1$ and $\varphi(x) = 0$ for $|x| \ge 2$, and we set $\omega_R(x) = \omega(x/R)$. For any $\varphi \in \hat{H}^1_{q',0}(\mathbb{R}^N_+)$ and for large $R > 0$, we have

$$(\mathrm{div} f, \varphi)_{\mathbb{R}^N_+} = (\omega_R \mathrm{div} f, \varphi)_{\mathbb{R}^N_+} = (\omega_R \Delta u, \varphi)_{\mathbb{R}^N_+} = -((\nabla\omega_R)\cdot(\nabla u), \varphi)_{\mathbb{R}^N_+} - (\omega_R \nabla \mathrm{u}, \nabla\phi)_{\mathbb{R}^N_+}. \tag{9.13}$$

By (9.12)

$$|((\nabla\omega_R)\cdot(\nabla u), \varphi)_{\mathbb{R}^N_+}| \le R^{-1}\|\nabla u\|_{L_q(D_{R,2R})}\|\varphi\|_{L_q(D_{R,2R})} \to 0$$

as $R \to \infty$. On the other hand, $(\mathrm{div} f, \varphi)_{\mathbb{R}^N_+} = -(\mathrm{f}, \nabla\varphi)_{\mathbb{R}^N_+}$, where we have used $\mathrm{f} \in C_0^\infty(\mathbb{R}^N_+)^N$. Thus, by (9.13) we have

$$(\nabla u, \nabla\varphi)_{\mathbb{R}^N_+} = (f, \nabla\varphi)_{\mathbb{R}^N_+}$$

for any $\varphi \in \hat{H}^1_{q',0}(\mathbb{R}^N_+)$. This shows that u is a solution of the weak Dirichlet problem. The uniqueness follows from the existence of solutions for the dual problem, which completes the proof of Theorem 9.3.

Regularity of the weak Dirichlet problem

In this appendix, we shall prove the following regularity theorem for the weak Dirichlet problem.

Theorem 10.1 Let $1 < q < \infty$. Let Ω be a uniform C^2 domain. Given $f \in L_q(\Omega)^N$, let $u \in \hat{H}^1_{q,0}(\Omega)$ be a unique solution of the weak Dirichlet problem (1.10) possessing the estimate: $\|\nabla u\|_{L_q(\Omega)} \le C\|\mathrm{f}\|_{L_q(\Omega)}$. If we assume that $\mathrm{div} f \in L_q(\Omega)$ in addition, then $\nabla^2 u \in L_q(\Omega)$ and

$$\|\nabla^2 u\|_{L_q(\Omega)} \le C_{M_2,q}(\|\mathrm{div} \mathrm{f}\|_{L_q(\Omega)} + \|\mathrm{f}\|_{L_q(\Omega)}).$$

Proof: Let ζ_j^i $(i = 0,1, j \in N)$ be cut-off functions given in Proposition 6.1. We first consider the regularity of $\zeta_j^0 u$. Let $c_j^0 = c_j^0(\phi)$ be a constant in Lemma 6.6 such that

$$\|u-c_j^0\|_{L_q(B_j^0)} \le c_1\|\nabla u\|_{L_q(B_j^0)}. \tag{10.1}$$

For any $\phi \in \hat{H}^1_{q'}(\mathbb{R}^N)$, we have

$$(\nabla(\zeta_j^0(u-c_j^0)),\nabla\phi)_{\mathbb{R}^N} = ((\nabla\zeta_j^0)(u-c_j^0),\nabla\phi)_{\mathbb{R}^N} + (\nabla u,\nabla(\zeta_j^0\phi))_{\mathbb{R}^N} - ((\nabla u)\cdot(\nabla\zeta_j^0),\phi)_{\mathbb{R}^N}$$
$$= -((\Delta\zeta_j^0)(u-c_j^0)+2(\nabla\zeta_j^0)\cdot(\nabla u)+\zeta_j^0 \mathrm{div}\, f,\phi)_{\mathbb{R}^N},$$

where we have used $(\nabla u,\nabla(\zeta_j^0\phi))_{\mathbb{R}^N} = (\mathrm{f},\nabla(\zeta_j^0\phi))_{\mathbb{R}^N} = -(\zeta_j^0\mathrm{div}\mathrm{f},\phi)_{\mathbb{R}^N}$.

Let $f=(\Delta\zeta_j^0)(u-c_j^0)+2(\nabla\zeta_j^0)\cdot(\nabla u)+\zeta_j^0\mathrm{div}\mathrm{f}$. Since $C_0^\infty(\mathbb{R}^N)\subset\hat{H}^1_q(\mathbb{R}^N)$, for any $\phi\in C_0^\infty(\mathbb{R}^N)$ we have

$(\Delta(\zeta_j^0(u-c_j^0)),\phi)_{\mathbb{R}^N} = (f,\phi)_{\mathbb{R}^N}$,

which yields that

$$\Delta(\zeta_j^0(u-c_j^0)) = f \quad \text{in } \mathbb{R}^N \tag{10.2}$$

in the sense of distribution. By Lemma 6.6, $f\in L_q(\mathbb{R}^N)$ and

$$\|f\|_{L_q(\mathbb{R}^N)} \le C(\|\mathrm{divf}\|_{L_q(B_j^0)} + \|\nabla u\|_{L_q(B_j^0)}). \tag{10.3}$$

From (10.2) it follows that

$$\partial_k\partial_\ell\Delta(\zeta_j^0(u-c_j^0)) = \partial_k\partial_\ell f$$

for any k, $\ell\in\mathbb{N}$. Since both sides are compactly supported distributions, we can apply the Fourier transform and the inverse Fourier transform. We then have

$$\partial_k\partial_\ell(\zeta_j^0(u-c_j^0)) = \mathcal{F}^{-1}\Big[\frac{\xi_k\xi_\ell}{|\xi|^2}\mathcal{F}[f](\xi)\Big].$$

By the Fourier multiplier theorem, we have

$$\|\partial_k\partial_\ell(\zeta_j^0(u-c_j^0))\|_{L_q(\mathbb{R}^N)} \le C\|f\|_{L_q(\mathbb{R}^N)}.$$

Since $\partial_k\partial_\ell(\zeta_j^0(u-c_j^0)) = \zeta_j^0\partial_k\partial_\ell u + (\partial_k\zeta_j^0)\partial_\ell u + (\partial_\ell\zeta_j^0)\partial_k u + (\partial_k\partial_\ell\zeta_j^0)(u-c_j^0)$, by (10.1) and (10.3) we have $\zeta_j^0\nabla^2 u\in L_q(\mathbb{R}^N)^N$ and

$$\|\zeta_j^0\nabla^2 u\|_{L_q(\Omega)} \le C_{M_2}(\|\mathrm{divf}\|_{L_q(B_j^0)} + \|\nabla u\|_{L_q(B_j^0)}). \tag{10.4}$$

We next consider $\zeta_j^1 u$. For any $\phi\in\hat{H}^1_{q',0}(\Omega_j)$, we have

$$(\nabla(\zeta_j^1 u),\nabla\phi)_{\Omega_J} = (g,\phi)_{\Omega_j}, \tag{10.5}$$

where we have set $g=-(\zeta_j^1\mathrm{divf}+2(\nabla u)\cdot(\nabla\zeta_j^1)+(\Delta\zeta_j^1)u)$. By Lemma 6.6, $g\in L_q(\Omega_j)$ and

$$\|g\|_{L_q(\Omega_j)} \le C(\|\mathrm{divf}\|_{L_q(\Omega\cap B_j^1)} + \|\nabla u\|_{L_q(\Omega\cap B_j^1)}). \tag{10.6}$$

We use the symbols given in Proposition 6.1. Let $a_{k\ell}$ and $b_{k\ell}$ be the $(k,\ell)^{th}$ component of $N\times N$ matrices A_j and B_j given in Proposition 6.1. By the change of variables: $y=\Phi_j(x)$, the variational equation (10.5) is transformed to

$$\sum_{k,\ell=1}^N((\delta_{k\ell}+A_{k\ell})\partial_k v,\partial_\ell\phi)_{\mathbb{R}^N_+} = (h,\phi)_{\mathbb{R}^N_+}. \tag{10.7}$$

Where, we have set

$$v=\zeta_j^1 u\circ\Phi_j, \quad h=g\circ\Phi_j, \quad J=\det(A_j+B_j)=1+J^0,$$

$$A_{k\ell} = \sum_{m=1}^N\{a_{\ell m}b_{km} + a_{\ell m}J_0(a_{k\ell}+b_{k\ell}) + b_{\ell m}J(a_{km}+b_{km})\}.$$

By Proposition 6.1 and (10.6), we have

$$\begin{aligned} \|A_{k\ell}\|_{L_\infty(\mathbb{R}^N)} &\le CM_1, \quad \|\nabla A_{k\ell}\|_{L_\infty(R^N)} \le C_K, \\ \|h\|_{L_q(\mathbb{R}^N_+)} &\le C(\|\mathrm{div} f\|_{L_q(B^1_j\cap\Omega)} + \|\nabla u\|_{L_q(B^1_j\cap\Omega)}), \end{aligned} \tag{10.8}$$

where C_K is a constant depending K, α, and β appearing in Definition 1.1. Since $supp\, f \subset \Phi^{-1}(B^1_j)\cap\mathbb{R}^N_+$, by Lemma 6.6

$$|(h,\phi)_{\mathbb{R}^N_+}| \le \|h\|_{L_q(B^1_j\cap\mathbb{R}^N_+)}\|\phi\|_{L_{q'}(B^1_j\cap\mathbb{R}^N_+)} \le C\|g\|_{L_q(B^1_j\cap\mathbb{R}^N_+)}\|\nabla\phi\|_{L_{q'}(\mathbb{R}^N_+)}$$

for any $\phi\in\hat H^1_{q',0}(\mathbb{R}^N_+)$, where C is a constant independent of $j\in\mathbb{N}$. Thus, by the Hahn-Banach theorem, there exists $h\in L_q(\mathbb{R}^N_+)^N$ such that $\|h\|_{L_q(\mathbb{R}^N_+)} \le C\|h\|_{L_q(B^1_j\cap\mathbb{R}^N_+)}$ and $(h,\nabla\phi)_{\mathbb{R}^N_+} = (h,\phi)_{\mathbb{R}^N_+}$ for any $\phi\in\hat H^1_{q',0}(\mathbb{R}^N_+)$. In particular, $\mathrm{div} h = -h\in L_q(\mathbb{R}^N_+)$. Thus, the variational problem (10.7) reads

$$\sum_{k,\ell=1}^N((\delta_{k\ell}+A_{k\ell})\partial_k v,\partial_\ell\phi)_{\mathbb{R}^N_+} = (h,\nabla\phi)_{\mathbb{R}^N_+} \quad \text{for any}\,\phi\in\hat{\mathrm{H}}^1_{\mathrm{q}',0}(\mathbb{R}^{\mathrm{N}}_+).$$

We now prove that if $M_1\in(0,1)$ is small enough, then for any $g\in L_q(\mathbb{R}^N_+)^N$, there exists a unique solution $w\in\hat H^1_{q,0}(\mathbb{R}^N_+)$ of the variational problem:

$$\sum_{k,\ell=1}^N((\delta_{k\ell}+A_{k\ell})\partial_k w,\partial_\ell\phi)_{\mathbb{R}^N_+} = (g,\nabla\phi)_{\mathbb{R}^N_+} \quad \text{forany}\,\phi\in\hat{\mathrm{H}}^1_{\mathrm{q}',0}(\mathbb{R}^{\mathrm{N}}_+). \tag{10.9}$$

having the estimate:

$$\|\nabla w\|_{L_q(\mathbb{R}^N_+)} \le C\|g\|_{L_q(\mathbb{R}^N_+)}. \tag{10.10}$$

Morevoer, if $\mathrm{div} g\in L_q(\mathbb{R}^N_+)$, then $\nabla w\in H^1_q(\mathbb{R}^N_+)^N$ and

$$\|\nabla^2 w\|_{L_q(\mathbb{R}^N_+)} \le C\|\mathrm{div} g\|_{L_q(\mathbb{R}^N_+)} + C_K\|g\|_{L_q(\mathbb{R}^N_+)}. \tag{10.11}$$

In fact, we prove the existence of w by the successive approximation. Let $w_1\in\hat H^1_{q,0}(\mathbb{R}^N_+)$ be a solution of the weak Dirichlet problem:

$$(\nabla w_1,\nabla\phi)_{\mathbb{R}^N_+} = (g,\nabla\phi)_{\mathbb{R}^N_+} \quad \text{forany}\,\phi\in\hat{\mathrm{H}}^1_{\mathrm{q}',0}(\mathbb{R}^{\mathrm{N}}_+). \tag{10.12}$$

By Theorem 9.3, w_1 uniquely exists and satisfies the estimate:

$$\|\nabla w_1\|_{L_q(\mathbb{R}^N_+)} \le C\|g\|_{L_q(\mathbb{R}^N_+)}. \tag{10.13}$$

Moreover, if we assume that $\mathrm{div} g\in L_q(\mathbb{R}^N_+)$ additionally, then $\nabla^2 w_1\in H^1_q(\mathbb{R}^N_+)^N$ and

$$\|\nabla^2 w_1\|_{L_q(\mathbb{R}^N_+)} \le C\|\mathrm{div} g\|_{L_q(\mathbb{R}^N_+)}. \tag{10.14}$$

Given $w_j\in\hat H^1_{q,0}(\mathbb{R}^N_+)$, let $w_{j+1}\in\hat H^1_{q,0}(\mathbb{R}^N_+)$ be a solution of the weak Dirichlet problem:

$$(\nabla w_{j+1},\nabla\phi)_{\mathbb{R}^N_+} = (g,\nabla\phi)_{\mathbb{R}^N_+} - \sum_{k,\ell=1}^N(A_{k\ell}\partial_k w_j,\partial_\ell\phi)_{\mathbb{R}^N_+} \quad \text{for any}\,\phi\in\hat{\mathrm{H}}^1_{\mathrm{q}',0}(\mathbb{R}^{\mathrm{N}}_+). \tag{10.15}$$

By Theorem 9.3 and (10.8), w_{j+1} exists and satisfies the estimate:

$$\|\nabla w_{j+1}\|_{L_q(\mathbb{R}^N_+)} \le C(\|g\|_{L_q(\Omega_+)} + M_1\|\nabla w_j\|_{L_q(\mathbb{R}^N_+)}). \tag{10.16}$$

Applying Theorem 9.3 and (10.8) to the difference $w_{j+1}-w_j$, we have

$$\|\nabla(w_{j+1}-w_j)\|_{L_q(\mathbb{R}^N_+)} \le CM_1\|\nabla(w_j-w_{j-1})\|_{L_q(\mathbb{R}^N_+)}. \tag{10.17}$$

Choosing $CM_1\le 1/2$ in (10.17), we see that $\{w_j\}_{j=1}^\infty$ is a Cauchy sequence in $\hat H^1_{q,0}(\mathbb{R}^N_+)$, and so the limit $w\in H^1_{q,0}(\mathbb{R}^N_+)$ exists and satisfies the weak Dirichlet problem (10.9). Moreover, taking the limit in (10.16), we have

$$\|\nabla w\|_{L_q(\mathbb{R}^N_+)} \le C\|g\|_{L_q(\Omega_+)} + CM_1\|\nabla w\|_{L_q(\mathbb{R}^N_+)}.$$

Since $CM_1\le 1/2$, we have $\|\nabla w\|_{L_q(\mathbb{R}^N_+)} \le 2C\|g\|_{L_q(\Omega_+)}$. Thus, we have proved that the weak Dirichlet problem (10.9) admits at least one solution $w \in \hat{H}^1_{q,0}(\Omega_+)$ possessing the estimate (10.10). The uniqueness follows from the existence of solutions to the dual problem. Thus, we have proved the unique existence of solutions of Eq.(10.9).

We now prove that $\nabla w \in H^1_q(\mathbb{R}^N_+)^N$ provided that $\operatorname{div} g \in L_q(\mathbb{R}^N_+)$. By Theorem 9.3, $\nabla w_1 \in H^1_q(\mathbb{R}^N_+)^N$ and w_1 satisfies the estimate:

$$\|\nabla^2 w_1\|_{L_q(\mathbb{R}^N_+)} \le C\|\operatorname{div} g\|_{H^1_q(\Omega_+)}.$$

Moreover, if we assume that $\nabla^2 w_j \in L_q(\mathbb{R}^N_+)^N$ in addition, then applying Theorem 9.3 to (10.15) and using (10.8), we see that $\nabla^2 w_{j+1} \in L_q(\mathbb{R}^N_+)^{N^2}$ and

$$\|\nabla^2 w_{j+1}\|_{L_q(\mathbb{R}^N_+)} \le C\|\operatorname{div} g\|_{L_q(\mathbb{R}^N_+)} + CM_1\|\nabla^2 w_j\|_{L_q(\mathbb{R}^N_+)} + C_K\|\nabla w_j\|_{L_q(\mathbb{R}^N_+)}. \tag{10.18}$$

And also, applying Theorem 9.3 to the difference $w_{j+1} - w_j$ and using (10.8), we have

$$\|\nabla^2(w_{j+1} - w_j)\|_{L_q(\mathbb{R}^N_+)} \le CM_1\|\nabla^2(w_j - w_{j-1})\|_{L_q(\mathbb{R}^N_+)} + C_K\|\nabla(w_j - w_{j-1})\|_{L_q(\mathbb{R}^N_+)},$$

which, combined with (10.17), leads to

$$\|\nabla^2(w_{j+1} - w_j)\|_{L_q(\mathbb{R}^N_+)} + \|\nabla(w_j - w_{j-1})\|_{L_q(\mathbb{R}^N_+)}$$

$$\le CM_1\|\nabla^2(w_j - w_{j-1})\|_{L_q(\mathbb{R}^N_+)} + C(C_K+1)M_1\|\nabla(w_{j-1} - w_{j-2})\|_{L_q(\mathbb{R}^N_+)}.$$

Choosing $M_1 > 0$ so small that $CM_1 \le 1/2$ and $(C_K+1)M_1 \le 1/2$, then we have

$$\|\nabla^2(w_{j+1} - w_j)\|_{L_q(\mathbb{R}^N_+)} + \|\nabla(w_j - w_{j-1})\|_{L_q(\mathbb{R}^N_+)} \le (1/2)^{j-1}L$$

with $L = \|\nabla^2(w_3 - w_2)\|_{L_q(\mathbb{R}^N_+)} + \|\nabla(w_2 - w_1)\|_{L_q(\mathbb{R}^N_+)}$. From this it follows that $\{\nabla^2 w_j\}_{j=1}^\infty$ is a Cauchy sequence in $L_q(\Omega)$, which yields that $\nabla^2 w \in L_q(\mathbb{R}^N_+)^{N^2}$. Moreover, taking the limit in (10.18) and using (10.10) gives that

$$\|\nabla^2 w\|_{L_q(\mathbb{R}^N_+)} \le C\|\operatorname{div} g\|_{L_q(\mathbb{R}^N_+)} + (1/2)\|\nabla^2 w\|_{L_q(\mathbb{R}^N_+)} + C_K\|g\|_{L_q(\mathbb{R}^N_+)}.$$

which leads to (10.11).

Applying what we have proved and using the estimate:

$$\|\operatorname{div} h\|_{L_q(\mathbb{R}^N_+)} + \|h\|_{L_q(\mathbb{R}^N_+)} \le C\|h\|_{L_q(\mathbb{R}^N_+)} \le C\|\operatorname{div} f\|_{L_q(B^1_j)} + \|\nabla u\|_{L_q(B^1_j)},$$

which follows from (10.8), we have $\nabla \mathrm{v} = \nabla(\zeta^1_j \mathrm{u} \circ \Phi_j) \in H^1_q(\mathbb{R}^N_+)^N$ and

$$\|\nabla(\zeta^1_j u \circ \Phi_j)\|_{H^1_q(\mathbb{R}^N_+)} \le C(\|\operatorname{div} f\|_{L_q(B^1_j \cap \Omega)} + \|\nabla u\|_{L_q(B^1_j \cap \Omega)}.$$

Since $\|u\|_{L_q(B^1_j \cap \Omega)} \le c_1\|\nabla u\|_{L_q(B^1_j \cap \Omega)}$ as follows from Lemma 6.6, we have

$$\|\zeta^1_j \nabla^2 \mathrm{u}\|_{L_q(\Omega)} \le C(\|\operatorname{div} \mathrm{f}\|_{L_q(B^1_j \cap \Omega)} + \|\nabla u\|_{L_q(B^1_j \cap \Omega)}). \tag{10.19}$$

Combining (10.4), (10.19) and (6.5) gives

$$\|\nabla^2 u\|_{L_q(\Omega)} \le C(\|\operatorname{div} \mathrm{f}\|_{L_q(\Omega)} + \|\nabla \mathrm{u}\|_{L_q(\Omega)}) \le C(\|\operatorname{div} \mathrm{f}\|_{L_q(\Omega)} + \|\mathrm{f}\|_{L_q(\Omega)}),$$

which completes the proof of Theorem 10.1.

A proof of Lemma 7.5

In this appendix, we shall prove Lemma 7.5. Namely, we prove the following lemma.

Lemma 11.1 Let $1 < q < \infty$ and let Ω be a uniformly C^2 domain whose inside is finite covering. Let O be a set given in Definition 7.1. Then, we have

$$\|\varphi\|_{L_q(O)} \le C_{q,S}\|\nabla\varphi\|_{L_q(\Omega)} \quad \text{for any } \varphi \in \hat{\mathrm{H}}^1_{q,0}(\Omega)$$

with some constant $C_{q,O}$ depending solely on O and q.

Proof: Let O_i ($i = 1, \ldots, \iota$) be the sub-domains given in Definition 7.1, and then it is sufficient to prove that

$$\|\varphi\|_{L_q(O_i)} \le C\|\nabla\varphi\|_{L_q(\Omega)} \quad \text{for any} \varphi \in \hat{\mathrm{H}}_{q,0}(\Omega) \text{and i=1,…,K.}$$

If $O_i \subset \Omega_R$ for some $R > 0$, since $\varphi|_\Gamma = 0$, by the usual Poincarés' inequality we have

$$\|\varphi\|_{L_q(O_i)} \le \|\varphi\|_{L_q(\Omega_R)} \le C\|\nabla\varphi\|_{L_q(\Omega_R)} \quad \text{for any} \varphi \in \hat{\mathrm{H}}^1_{q,0}(\Omega).$$

Let O_i be a subdomain for which the condition b in Definition 7.1 holds. Since the norms for $\varphi(A \circ \tau(y))$ and $\varphi(y)$ are equivalent, without loss of generality we may assume that

$$O_i \subset \{x = (x', x_N) \in \mathbb{R}^N \mid a(x') < x_N < b, \quad x' \in D\} \subset \Omega$$
$$\{x = (x', x_N) \in \mathbb{R}^N \mid x_N = a(x') \quad x' \in D\} \subset \Gamma.$$

Since $\varphi \in \hat{H}^1_{q,0}(\Omega)$, we can write

$$\varphi(x', x_N) = \int_{a(x')}^{x_N} (\partial_s\varphi)(x', s)ds.$$

because $\varphi(x', a(x')) = 0$. By Hardy inequality (9.11), we have

$$\left(\int_{a(x')}^{b} |\varphi(x', x_N)|^q x_N^{-q} dx_N\right)^{1/q} \le \left(\int_{a(x')}^{b}\left(\int_{a(x')}^{x_N} |(\partial_s\varphi)(x', s)|ds\right)^q x_N^{-q} dx_N\right)^{1/q}$$
$$\le \frac{q}{q+1}\left(\int_{a(x')}^{b} |s\partial_s\varphi(x', s)|^q s^{-q} ds\right)^{1/q},$$

and so, by Fubini's theorem we have

$$\left(\int_{O_i} |\varphi(x)|^q dx\right)^{1/q} \le \left(\int_D dx' \int_{a(x')}^{b} |\varphi(x', x_N)|^q dx_N\right)^{1/q}$$
$$\le \left(\int_D dx' \int_{a(x')}^{b} |\varphi(x', x_N)|^q x_N^{-q} b^q dx_N\right)^{1/q} \le \frac{qb}{q+1}\left(\int_D dx' \int_{a(x')}^{b} |\partial_N\varphi(x)|^q dx_N\right)^{1/q}$$
$$\le \frac{qb}{q+1}\|\nabla\varphi\|_{L_q(\Omega)}.$$

This completes the proof of Lemma 11.1.

Remark on a proof of Proposition 6.1

In Enomoto-Shibata[37] [Appendix], instead of $\left\|\nabla(B^i_j, B^i_{j,-})\right\|_{L_\infty(\mathbb{R}^N)} \le C_K$, it was proved that

$$\left\|\nabla(B^i_j, B^i_{j,-})\right\|_{L_\infty(\mathbb{R}^N)} \le CM_2,$$

that is the estimate of $\nabla(B^i_j, B^i_{j,-})$ depends on M_1. We shall give an idea how to improve this point below. Let $x_0 = (x_{0'}, x_{0N}) \in \Gamma$ and we assume that

$$\Omega \cap B_\beta(x_0) = \{x \in \mathbb{R}^N \mid x_N > h(x')(x' \in B'_\alpha(x_{0'}))\} \cap B_\beta(x_0),$$
$$\Gamma \cap B_\beta(x_0) = \{x \in \mathbb{R}^N \mid x_N = h(x')(x' \in B'_\alpha(x_{0'}))\} \cap B_\beta(x_0).$$

We only consider the case where $k = 3$. In fact, by the same argument, we can improve the estimate in the case where $k = 2$. We assume that $h \in C^3(B'_\alpha(x_{0'}))$, $\|h\|_{H^3_\infty(B'_\alpha(x_{0'}))} \le K$, and $x_{0N} = h(x_{0'})$. Below, C denotes a generic constant depending on K, α and β but independent of ε. Let $\rho(y)$ be a function in $C_0^\infty(\mathbb{R}^N)$ such that $\rho(y) = 1$ for $|y'| \le 1/2$ and $|y_N| \le 1/2$ and $\rho(y) = 0$ for $|y'| \ge 1$ or $|y_N| \ge 1$. Let $\rho_\varepsilon(y) = \rho(y/\varepsilon)$. We consider the C^∞ diffeomorphism:;

$$x_j = \Phi^\varepsilon_j(y) = x_{0j} + \sum_{k=1}^{N} t_{j,k} y_k + \sum_{k,\ell=1}^{N} s_{j,k\ell} y_k y_\ell \rho_\varepsilon(y).$$

Where, $t_{j,k}$ and $s_{j,k\ell}$ are some constants satisfying the conditions (12.3), (12.1), and (12.2), below. Let

$$G_\varepsilon(y) = \Phi^\varepsilon_N(y) - h(\Phi^\varepsilon_1(y), \ldots, \Phi^\varepsilon_{N-1}(y)).$$

Notice that $G_\varepsilon(0) = x_{0N} - h(x'_0) = 0$. We choose $t_{j,k}$ and $s_{\ell,mn}$ in such a way that

$$\frac{\partial G_\varepsilon}{\partial y_N}(0) = t_{N,N} - \sum_{k=1}^{N-1} \frac{\partial h}{\partial x_k}(x_{0'})t_{k,N} \ne 0, \quad \frac{\partial G_\varepsilon}{\partial y_j}(0) = t_{N,j} - \sum_{k=1}^{N-1} \frac{\partial h}{\partial x_k}(x_{0'})t_{k,j} = 0, \qquad (12.1)$$

$$\frac{\partial^2 G_\varepsilon}{\partial y_\ell \partial y_m}(0) = s_{N,\ell m} + s_{N,m\ell} - \sum_{k=1}^{N-1} \frac{\partial h}{\partial x_k}(x_{0'})(s_{k,\ell m} + s_{k,m\ell}) - \sum_{j,k=1}^{N-1} \frac{\partial^2 G_\varepsilon}{\partial x_j \partial x_k}(x_0')t_{j,\ell}t_{k,m} = 0. \tag{12.2}$$

Moreover, setting

$$T = \begin{pmatrix} t_{1,1} & t_{2,1} & \cdots & t_{N,1} \\ t_{1,2} & t_{2,2} & \cdots & t_{N,2} \\ \vdots & \vdots & \ddots & \vdots \\ t_{1,N} & t_{2,N} & \cdots & t_{N,N} \end{pmatrix}$$

we assume that T is an orthogonal matrix, that is

$$\sum_{\ell=1}^N t_{\ell,m}t_{\ell,n} = \delta_{mn} = \begin{cases} 1 & \text{for m=n,} \\ 0 & \text{for m}\neq\text{n.} \end{cases} \tag{12.3}$$

We write $\frac{\partial h}{\partial x_j}(x_{0'})$ simply by h_j and set $H_j = \sqrt{1+\sum_{\ell=j}^{N-1} h_\ell^2} = \sqrt{1+h_j^2+h_{j+1}^2+\cdots+h_{N-1}^2}$. Let

$$t_{N,N-j} = \frac{h_{N-j}}{H_{N-j}H_{N+1-j}}, \quad t_{N-k,N-j} = -\frac{h_{N-k}h_{N-j}}{H_{N-j}H_{N+1-j}}(k=1,\dots,j-1),$$

$$t_{N-j,N-j} = \frac{H_{N+1-j}}{H_{N-1}}, \quad t_{k,N-j} = 0(k=1,\dots,N-j-1)$$

$$\text{for j=1,}\dots\text{,N}-1, \text{ and } t_{i,N} = -\frac{h_i}{H_1}(i=1,\dots,N-1), \quad t_{N,N} = \frac{1}{H_1}.$$

Then, we see that such $t_{j,k}$ satisfy (12.1) and (12.3). In particular,

$$\frac{\partial G_\varepsilon}{\partial y_N}(0) = \frac{1}{H_1}. \tag{12.4}$$

Moreover, assuming the symmetry: $s_{\ell,jk} = s_{\ell,kj}$, we have

$$s_{N,jk} = \frac{1}{2H_2}\sum_{m,n=1}^{N-1} \frac{\partial^2 h}{\partial x_m \partial x_n}(x_{0'})t_{m,j}t_{n,k}, \quad s_{i,jk} = -\frac{h_i}{2H_1^2}\sum_{m,n=1}^{N-1} \frac{\partial^2 h}{\partial x_m \partial x_n}(x_{0'})t_{m,j}t_{n,k}.$$

By successive approximation, we see that there exists a constant $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$ there exists a function $\psi_\varepsilon \in C^3(B'_\varepsilon(0))$ satisfies the following conditions:

$$\psi_\varepsilon(0) = \partial_i\psi_\varepsilon(0) = \partial_i\partial_j\psi_\varepsilon(0) = 0,$$

$$\|\psi_\varepsilon\|_{L_\infty(B'_\varepsilon(0))} \le C\varepsilon^2, \|\partial_i\psi_\varepsilon\|_{L_\infty(B'_\varepsilon(0))} \le C\varepsilon, \|\partial_i\partial_j\psi_\varepsilon\|_{L_\infty(B'_\varepsilon(0))} \le C, \|\partial_i\partial_j\partial_k\psi_\varepsilon\|_{L_\infty(B'_\varepsilon(0))} \le C\varepsilon^{-1},$$

$$G_\varepsilon(y',\psi_\varepsilon(y')) = 0 \quad \text{for } y' \in B'_\varepsilon(0), \tag{12.5}$$

where i , j and k run from 1 through $N-1$. Notice that

$$x_N - h_\varepsilon(x') = G_\varepsilon(y) = G_\varepsilon(y',\psi_\varepsilon(y')) + \int_0^1 (\partial_N G_\varepsilon)(y',\psi_\varepsilon(y') + \theta(y_N - \psi_\varepsilon(y')))d\theta(y_N - \psi_\varepsilon(y'))$$
$$= (\partial_N G_\varepsilon)(0) + \tilde G_\varepsilon(y))(y_N - \psi_\varepsilon(y')), \tag{12.6}$$

where we have used $G_\varepsilon(y',\psi_\varepsilon(y')) = 0$ and we have set

$$\tilde G_\varepsilon(y) = \int_0^1\int_0^1 \{\sum_{\ell=1}^{N-1} (\partial_\ell\partial_N G_\varepsilon)(\tau y', \tau(\psi_\varepsilon(y') + \theta(y_N - \psi_\varepsilon(y')))y_\ell$$

$$+\partial_N^2 G_\varepsilon(\tau y', \tau(\psi_\varepsilon(y') + \theta(y_N - \psi_\varepsilon(y')))(\psi_\varepsilon(y') + \theta(y_N - \psi_\varepsilon(y')))\} d\theta d\tau.$$

Since $(\partial_N G_\varepsilon)(0) = 1/H_1$, choosing $\varepsilon_0 > 0$ so small that $|\tilde G_\varepsilon(y)| \le 1/(2H_1)$ for $|y| \le \varepsilon_0$, we see that $x_N - h(x') \ge 0$ and $y_N - \psi_\varepsilon(y') \ge 0$ are equivalent.

Let ω be a function in $C_0^\infty(\mathbb{R}^{N-1})$ such that $\omega(y') = 1$ for $|y'| \le 1/2$ and $\omega(y') = 0$ for $|y'| \ge 1$ and set $\omega_\varepsilon(y') = \psi_\varepsilon(y')\omega(y'/\varepsilon)$. Then, by (12.5) we have

$$\begin{aligned} \|\omega_\varepsilon\|_{L_\infty(\mathbb{R}^{N-1})} &\le C\varepsilon^2, \quad {}_{L_\infty(\mathbb{R}^{N-1})} \le C\varepsilon, \|\partial_i\omega_\varepsilon\| \\ \|\partial_i\partial_j\omega_\varepsilon\|_{L_\infty(\mathbb{R}^{N-1})} &\le C\varepsilon, \quad \|\partial_i\partial_j\partial_k\omega_\varepsilon\|_{L_\infty(\mathbb{R}^{N-1})} \le C\varepsilon^{-1}. \end{aligned} \tag{12.7}$$

where i, j, and k run from 1 through $N-1$. Setting $\Psi^\varepsilon(z) = \Phi^\varepsilon(z', z_N + \omega_\varepsilon(z'))$, that is $y_N = z_N + \omega_\varepsilon(z')$, and $y_j = z_j$ for $j = 1, \ldots, N-1$, we see that there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, the map: $z \to x = \Psi^\varepsilon(z)$ is a diffeomorphism of C^3 class from $\mathbb{R}^N$ onto $\mathbb{R}^N$. Since

$$\frac{\partial x_m}{\partial z_k} = t_{m,k} + b_{m,k}, \quad \frac{\partial x_m}{\partial z_N} = t_{m,N} + b_{n,N}$$

where we have set

$$b_{m,k} = \sum_{i,j=1} \frac{\partial}{\partial z_k}(s_{m,ij} y_i y_j \rho_\varepsilon(y)) + \{t_{m,N} + \sum_{i,j=1}^{N} \frac{\partial}{\partial z_k}(s_{m,ij} y_i y_j \rho_\varepsilon(y))\} \frac{\partial \omega_\varepsilon}{\partial z_k}(z'),$$

$$b_{n,N} = \sum_{i,j=1} \frac{\partial}{\partial z_N}(s_{m,ij} y_i y_j \rho_\varepsilon(y)),$$

let A and B be the $N \times N$ matrices whose $(m,n)^{th}$ components are $t_{m,n}$ and $b_{m,n}$, respectively. Then, by (12.7), A is an orthogonal matrix and B satisfies the estimates:

$$\|B\|_{L_\infty(\mathbb{R}^N)} \le C\varepsilon, \quad \|\nabla B\|_{L_\infty(\mathbb{R}^N)} \le C, \quad \|\nabla^2 B\|_{L_\infty(\mathbb{R}^N)} \le C\varepsilon^{-1}.$$

Moreover, by (12.6) we have

$$x_N - h(x') = (1/H_1 + \tilde{G}_\varepsilon(z', z_N + \omega_\varepsilon(z')))(z_N + (\omega(z'/\varepsilon) - 1)\psi_\varepsilon(z')),$$

which shows that when $|z'| \le \varepsilon/2$, $x_N \ge h(x')$ and $z_N \ge 0$ are equivalent. We can construct the sequences of $C_0^\infty(\mathbb{R}^N)$ functions, $\{\zeta_j^i\}$, $\{\tilde{\zeta}_j^i\}$, by standard manner (cf. Shibata et al.[37] [Appendix]). This completes the proof of Proposition 6.1.[46–48]

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